Please note that

- **Due date:April 16, Tuesday, by 11:55 pm.**
- Late submission is **NOT** accepted.

Question 1. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation.

1. Show that $S : \mathbb{R}^n \to \mathbb{R}^m$ defined as $S(x) = L(x) + T(x)$ is a linear transformation.
2. Let A and B be the matrix representations of L and T respectively. Verify that $A + B$ is the matrix representation of S.

Question 2. Consider two linear transformations $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^p$. Define $L : \mathbb{R}^n \to \mathbb{R}^p$ as $L(x) = T(S(x)), x \in \mathbb{R}^n$.

1. Show that L is a linear transformation.
2. Let A and B be the matrix representation of S and T respectively.
   a) Justify why the matrix product BA is defined.
   b) Verify that BA is the matrix representation of L.

Question 3. Consider $A, B \in \mathbb{R}^{n\times n}$. Suppose that A and B are similar.

1. Show that A^2 and B^2 are similar.
2. Show that, for any integer $n \geq 3$, A^n and B^n are similar.
3. Show that, if $A^2 = I$, then $B^2 = I$.
4. Show that, if A is invertible, then B is invertible and A^{-1} and B^{-1} are similar.

5. Show that there is an invertible matrix S such that $Null(B) = \{x \in \mathbb{R}^n | \ \exists y \in Null(A), x = S^{-1}y\}$.
6. Show that $n(A) = n(B)$ and $rank(A) = rank(B)$.

Question 4. Consider the linear transformation $L: P_3 \to P_2$ defined as

$$L(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + (2a_2 - a_1 - a_0)x + (2a_3 - 2a_1 - a_0)x^2.$$

1. Find A, the matrix of L in the bases $B = \{1, x, x^2, x^3\}$ and $B' = \{1, x, x^2\}$.
2. Verify that, if $p \in Ker(L)$, then $x = [p]_B \in Null(A)$.
3. Find a basis of $Ker(L)$.
4. Verify that, if $q \in L(P_3)$, then $y = [q]_{B'} \in Col(A)$.
5. Find a basis of $L(P_3)$.

Question 5. Are the following mappings linear transformations?

a) $L: \mathbb{R}^2 \to \mathbb{R}^3,\ L(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ |x_1 + x_2| \end{bmatrix}$.

b) $L: P_2 \to \mathbb{R}^2,\ L(a_0 + a_1 x + a_2 x^2) = \begin{bmatrix} a_0 a_1 \\ a_2 \end{bmatrix}$.

Question 6. Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $L(x) = Ax$, where $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$.

Consider the basis of $\mathbb{R}^3$, $B = \{v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\}$. Let $B_0 = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$.

1. Find the transition matrix from B to B_0.
2. Find by two different methods, the matrix of L with respect to the basis B.