p	q	r	$p \wedge q$	$(p \wedge q) \supset r$
true	true	true	true	true
true	true	false	true	false
true	false	true	false	true
true	false	false	false	true
false	true	true	false	true
false	true	false	false	true
false	false	true	false	true
false	false	false	false	true

Table 2.1: Truth table for the propositional formula $(p \wedge q) \supset r$

The key to answering these questions for any logic is to have rigorous, mathematical semantics that define precisely what a given statement means. These formal semantics provide a basis by which one can independently assess the trustworthiness of a logical system. For example, in propositional logic, the formal meaning of a statement such as

$$(p \wedge q) \supset r$$

can be calculated using a truth table, as illustrated in Table 2.1. Each line in the truth table corresponds to a particular interpretation of the propositional variables (i.e., a mapping of variables to specific truth values). Truth tables calculate the meaning of larger formulas in a *syntax-directed* way, based on the meanings of their components: for example, the meaning of $(p \wedge q) \supset r$ for a given interpretation is calculated using the meanings of $p \wedge q$ and r, as well as a specific rule for the operator $\supset$. A propositional-logic formula is a *tautology*—and therefore safe to use as an axiom of the system—if it is true for *every* possible interpretation of the propositional variables.

These same core ideas apply to the semantics for our access-control logic. Because we must account for the interpretation of *principals* in addition to *propositional variables*, the semantics requires a little more structure than truth tables provide. We can find this additional structure in the form of *Kripke structures*.

2.3.1 Kripke Structures

Kripke structures are useful models for analyzing a variety of situations. They are commonly used to provide semantics for modal and temporal logics, providing a basis for automated model checking.

Definition 2.1 *A* Kripke structure $\mathcal{M}$ *is a three-tuple* $\langle W, I, J \rangle$*, where:*

- W *is a nonempty set, whose elements are called* worlds*.*
- $I : \textbf{\textit{PropVar}} \to \mathcal{P}(W)$ *is an* interpretation *function that maps each propositional variable to a set of worlds.*

- *J : **PName** $\rightarrow \mathcal{P}(W \times W)$ is a function that maps each principal name to a relation on worlds (i.e., a subset of $W \times W$).* ∎

Before we look at some examples of Kripke structures, a few comments about this definition are in order. First, the concept of *worlds* is an abstract one. In reality, W is simply a set: its contents (whatever they may be) are called worlds. In many situations, the notion of worlds corresponds to the notion of system states or to the concept of possible alternatives.

Second, the functions I and J provide meanings (or *interpretations*) for our propositional variables and simple principals. These meanings will form the basis for our semantics of arbitrary formulas in our logic. Intuitively, $I(p)$ is the set of worlds in which we consider p to be true. $J(A)$ is a relation that describes how the simple principal A views the relationships between worlds: each pair $(w,w') \in J(A)$ indicates that, when the current world is w, principal A believes it possible that the current world is w'.

For illustration purposes, we introduce some examples of Kripke structures. The first example provides some intuition as to what the interpretation functions I and J represent, illustrating how each relation $J(P)$ might reflect principal P's understanding of the universe.

Example 2.7

Consider the situation of three young children (*Flo*, *Gil*, and *Hal*), who are being looked after by an overprotective babysitter. This babysitter will let them go outside to play only if the weather is both sunny and sufficiently warm.

To keep things simple, let us imagine that there are only three possible situations: it is sunny and warm, it is sunny but cool, or it is not sunny. We can represent these possible alternatives with a set of three worlds: $W_0 = \{sw, sc, ns\}$.

We use the propositional variable g to represent the proposition "The children can go outside." The baby sitter's overprotectiveness can be represented by any interpretation function

$$I_0 : \textbf{PropVar} \rightarrow \mathcal{P}(\{sw, sc, ns\})$$

for which $I_0(g) = \{sw\}$. That is, the proposition g ("the children can go outside") is true only in the world sw (i.e., when the weather is both sunny and warm).

Now, the children themselves are standing by the window, trying to determine whether or not they'll be allowed to go outside. *Gil*, who is tall enough to see the outdoor thermometer, possesses perfect knowledge of the situation, as he will be able to determine whether it is both sunny and sufficiently warm. This perfect knowledge corresponds to a possible-worlds relation

$$J_0(Gil) = \{(sw,sw), (sc,sc), (ns,ns)\}.$$

Whatever the current situation is, *Gil* has the correct understanding of the situation. (Note that $J_0(Gil)$ is the identity relation id_{W_0} over the set W_0.)

In contrast, *Flo* is too short to see the outdoor thermometer, and thus she cannot distinguish between the "sunny and warm" and "sunny and cool" alternatives. This uncertainty corresponds to a possible-worlds relation

$$J_0(Flo) = \{(sw,sw),(sw,sc),(sc,sw),(sc,sc),(ns,ns)\}.$$

Thus, for example, if the current situation is "sunny and warm" (i.e., *sw*), *Flo* considers both "sunny and warm" and "sunny and cool" as legitimate possibilities. That is, $J_0(Flo)(sw) = \{sw,sc\}$.

Finally, *Hal* is too young to understand that it can be simultaneously sunny and cool: he believes that the presence of the sun automatically makes it warm outside. His confusion corresponds to a possible-worlds relation

$$J_0(Hal) = \{(sw,sw),(sc,sw),(ns,ns)\}.$$

Whenever the actual weather is sunny and cool, *Hal* believes it to be sunny and warm: $J_0(Hal)(sc) = \{sw\}$.

The tuple $\langle W_0, I_0, J_0\rangle$ forms a Kripke structure. ◊

The next example introduces a Kripke structure that does not necessarily reflect any particular scenario or vignette. Rather, the Kripke structure is merely a three-tuple that contains a set and two functions that match the requirements of Definition 2.1.

Example 2.8

Let $W_1 = \{w_0, w_1, w_2\}$ be a set of worlds, and let $I_1 : \mathbf{PropVar} \to \mathcal{P}(W_1)$ be the interpretation function defined as follows[2]:

$$\begin{aligned} I_1(q) &= \{w_0, w_2\}, \\ I_1(r) &= \{w_1\}, \\ I_1(s) &= \{w_1, w_2\}. \end{aligned}$$

In addition, let $J_1 : \mathbf{PName} \to \mathcal{P}(W_1 \times W_1)$ be the function defined as follows[3]:

$$\begin{aligned} J_1(Alice) &= \{(w_0,w_0),(w_1,w_1),(w_2,w_2)\}, \\ J_1(Bob) &= \{(w_0,w_0),(w_0,w_1),(w_1,w_2),(w_2,w_1)\}. \end{aligned}$$

The three-tuple $\langle W_1, I_1, J_1\rangle$ is a Kripke structure. Intuitively, proposition q is true in worlds w_0 and w_2, r is true in world w_1, and s is true in worlds w_1 and w_2. All other propositions are false in all worlds. ◊

[2]In this example and those that follow, we adopt the convention of specifying only those propositional variables that the interpretation function maps to nonempty sets of worlds. Thus, for any propositional variable p not explicitly mentioned, we assume that $I_1(p) = \emptyset$.

[3]We adopt a similar convention for principal-mapping functions J: for any principal name A for which $J(A)$ is not explicitly defined, we assume that $J(A) = \emptyset$.

Present State	*Next State* $x = 0$	$x = 1$
A	*A*	*D*
B	*A*	*C*
C	*C*	*B*
D	*C*	*A*

Table 2.2: State-transition table for finite-state machine M

World	p	q	r	s
A	true	true	false	true
B	false	true	false	true
C	true	false	false	true
D	false	true	false	true

Table 2.3: Truth values of primitive propositions *p, q, r,* and *s* in each world

The next example illustrates how a Kripke structure might be used to represent a state machine.

Example 2.9
Consider the state-transition table for a finite-state machine M shown in Table 2.2. This machine has four states: A, B, C, and D. The column labeled "Present State" lists the possible *present states* of M. The two columns under the label "Next State" list the *next states* of M if the input x is either 0 or 1, respectively. For example, the second row of Table 2.2 describes M's behavior whenever it is currently in state B: if the input is x is 0, then the next state will be A; if x is 1, then the next state will be C.

We can construct a Kripke structure $\langle W_2, I_2, J_2 \rangle$ to model this machine by defining W_2 to be the set of M's states:

$$W_2 = \{A, B, C, D\}.$$

Now, suppose that there are four primitive propositions (p, q, r, s) associated with the state machine M, with their truth values in the various states given by Table 2.3. This table effectively specifies the interpretation function I_2 on these propositions, namely:

$$\begin{aligned} I_2(p) &= \{A, C\}, \\ I_2(q) &= \{A, B, D\}, \\ I_2(r) &= \{\}, \\ I_2(s) &= \{A, B, C, D\}. \end{aligned}$$

Finally, imagine that there is an observer *Obs* of the machine's execution. This observer has faulty knowledge of M's states: whenever M is in state C, *Obs* incorrectly believes M to be in state D. We'll assume that the observer *does* correctly know when M is in states A, B, or D.

This observer's state knowledge can be captured by the following relation:

$$J_2(Obs) = \{(A,A),(B,B),(C,D),(D,D)\}.$$

In the relation $J_2(Obs)$, the first element of each pair represents the present state of M, and the second element is the observed state of M. Thus the pair (C,D) reflects that, whenever M is in state C, Obs always believes the current state is D.

The tuple $\langle\{A,B,C,D\},I_2,J_2\rangle$ forms a Kripke structure. ◊

In the next example, we model the same state machine, but we consider the inputs (i.e., "x=0" or "x=1") as the "observers" of the state machine, and we use each "next state" as the perceived state by the particular observer. Although the set of worlds and the interpretation function do not change, the principal-mapping function does change.

Example 2.10

Let W_2 and I_2 be as defined in Example 2.9, and define J_3 as follows:

$$\begin{aligned} J_3(X_0) &= \{(A,A),(B,A),(C,C),(D,C)\},\\ J_3(X_1) &= \{(A,D),(B,C),(C,B),(D,A)\}. \end{aligned}$$

The tuple $\langle\{A,B,C,D\},I_2,J_3\rangle$ forms a Kripke structure. ◊

Just as the interpretation function I of a Kripke structure provides the base interpretation for propositional variables, the function J provides a base interpretation for simple principal names. We extend J to work over arbitrary *principal expressions*, using set union and relational composition as follows:

$$\begin{aligned} J(P \mathbin{\&} Q) &= J(P) \cup J(Q),\\ J(P \mid Q) &= J(P) \circ J(Q). \end{aligned}$$

Example 2.11

Suppose that we have the following relations:

$$\begin{aligned} J(Andy) &= \{(w_0,w_0),(w_0,w_2),(w_1,w_1),(w_2,w_1)\},\\ J(Stu) &= \{(w_1,w_2)\},\\ J(Keri) &= \{(w_0,w_2),(w_1,w_2),(w_2,w_2)\}. \end{aligned}$$

Then $J(Keri \mid (Andy \mathbin{\&} Stu))$ is calculated as follows:

$$\begin{aligned} &J(Keri \mid (Andy \mathbin{\&} Stu))\\ &\quad = J(Keri) \circ J(Andy \mathbin{\&} Stu),\\ &\quad = J(Keri) \circ (J(Andy) \cup J(Stu)),\\ &\quad = J(Keri) \circ \{(w_0,w_0),(w_0,w_2),(w_1,w_1),(w_2,w_1),(w_1,w_2)\}\\ &\quad = \{(w_0,w_1),(w_1,w_1),(w_2,w_1)\}. \end{aligned}$$

◊

✎ **Exercise 2.3.1** *Recall the Kripke structure* $\langle W_0, I_0, J_0 \rangle$ *from Example 2.7, and further suppose that*

$$J_0(Ida) = \{(sw, sc), (sc, sw), (ns, sc), (ns, ns)\}.$$

Calculate the following relations:

a. $J_0(Hal \,\&\, Gil)$

b. $J_0(Gil \mid Hal)$

c. $J_0(Flo \,\&\, Ida)$

d. $J_0(Hal \mid Ida)$

e. $J_0(Ida \mid Hal)$

f. $J_0(Hal \,\&\, (Ida \mid Hal))$

g. $J_0(Hal \mid (Ida \,\&\, Hal))$

2.3.2 Semantics of the Logic

The Kripke structures provide the foundation for a formal, precise, and rigorous interpretation of formulas in our logic. For each Kripke structure $\mathcal{M} = \langle W, I, J \rangle$, we can define what it means for formulas in our logic to be *satisfied* in the structure. We can also identify those worlds in W for which a given formula is said to be true.

To define the semantics, we introduce a family of *evaluation functions*. Each Kripke structure $\mathcal{M} = \langle W, I, J \rangle$ gives rise to an evaluation function $\mathcal{E}_\mathcal{M}$ that maps well-formed formulas in the logic to subsets of W. Intuitively, $\mathcal{E}_\mathcal{M}[\![\varphi]\!]$ is the set of worlds from the Kripke structure $\mathcal{M}$ for which the well-formed formula φ is considered true. We say that $\mathcal{M}$ *satisfies* φ (written $\mathcal{M} \models \varphi$) whenever φ is true in *all* of the worlds of $\mathcal{M}$: that is, when $\mathcal{E}_\mathcal{M}[\![\varphi]\!] = W$. It follows that a Kripke structure $\mathcal{M}$ *does not satisfy* φ (written $\mathcal{M} \not\models \varphi$) when there exists at least one $w \in W$ such that $w \notin \mathcal{E}_\mathcal{M}[\![\varphi]\!]$.

Each $\mathcal{E}_\mathcal{M}$ is defined inductively on the structure of well-formed formulas, making use of the interpretation functions I and J within the Kripke structure $\mathcal{M} = \langle W, I, J \rangle$. We discuss the individual cases separately, starting with the standard propositional operators and then moving on to the access-control specific cases. The full set of definitions is also summarized in Figure 2.1.

Standard Propositional Operators

The semantics for propositional variables and the standard logical connectives (e.g., negation, conjunction, implication) are very similar to the truth-table interpretations for standard propositional logic. The interpretation function I identifies those worlds in which the various propositional variables are true, while the semantics of the other operators are defined using standard set operations. We handle these cases in turn.

FIGURE 2.1 Semantics of core logic, for each $\mathcal{M} = \langle W, I, J \rangle$

$$\begin{aligned}
\mathcal{E}_{\mathcal{M}}[\![p]\!] &= I(p)\\
\mathcal{E}_{\mathcal{M}}[\![\neg\varphi]\!] &= W - \mathcal{E}_{\mathcal{M}}[\![\varphi]\!]\\
\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \wedge \varphi_2]\!] &= \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!] \cap \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!]\\
\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \vee \varphi_2]\!] &= \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!] \cup \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!]\\
\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \supset \varphi_2]\!] &= (W - \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!]) \cup \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!]\\
\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \equiv \varphi_2]\!] &= \mathcal{E}_{\mathcal{M}}[\![\varphi_1 \supset \varphi_2]\!] \cap \mathcal{E}_{\mathcal{M}}[\![\varphi_2 \supset \varphi_1]\!]\\
\mathcal{E}_{\mathcal{M}}[\![P \Rightarrow Q]\!] &= \begin{cases} W, & \text{if } J(Q) \subseteq J(P)\\ \emptyset, & \text{otherwise}\end{cases}\\
\mathcal{E}_{\mathcal{M}}[\![P \textsf{ says } \varphi]\!] &= \{w | J(P)(w) \subseteq \mathcal{E}_{\mathcal{M}}[\![\varphi]\!]\}\\
\mathcal{E}_{\mathcal{M}}[\![P \textsf{ controls } \varphi]\!] &= \mathcal{E}_{\mathcal{M}}[\![(P \textsf{ says } \varphi) \supset \varphi]\!]
\end{aligned}$$

Propositional Variables: The truth of a propositional variable p is determined by the interpretation function I: a variable p is considered true in world w precisely when $w \in I(p)$. Thus, for all propositional variables p,

$$\mathcal{E}_{\mathcal{M}}[\![p]\!] = I(p).$$

For example, if $\mathcal{M}_0$ is the Kripke structure $\langle W_0, I_0, J_0 \rangle$ from Example 2.7, $\mathcal{E}_{\mathcal{M}_0}[\![g]\!] = I_0(g) = \{sw\}$.

Negation: A formula with form $\neg\varphi$ is true in precisely those worlds in which φ is *not* true. Because (by definition) $\mathcal{E}_{\mathcal{M}}[\![\varphi]\!]$ is the set of worlds in which φ is true, we define

$$\mathcal{E}_{\mathcal{M}}[\![\neg\varphi]\!] = W - \mathcal{E}_{\mathcal{M}}[\![\varphi]\!].$$

Thus, returning to Example 2.7,

$$\mathcal{E}_{\mathcal{M}_0}[\![\neg g]\!] = W_0 - \mathcal{E}_{\mathcal{M}_0}[\![g]\!] = \{sw, sc, ns\} - \{sw\} = \{sc, ns\}.$$

Notice that $\mathcal{E}_{\mathcal{M}_0}[\![\neg g]\!]$ is the set of worlds in which the children are *not* allowed to go outside.

Conjunction: A conjunctive formula $\varphi_1 \wedge \varphi_2$ is considered true in those worlds for which *both* φ_1 and φ_2 are true: that is, $\varphi_1 \wedge \varphi_2$ is true in those worlds w for which $w \in \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!]$ *and* $w \in \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!]$. Thus, we can define $\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \wedge \varphi_2]\!]$ in terms of set intersection:

$$\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \wedge \varphi_2]\!] = \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!] \cap \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!].$$

Disjunction: Likewise, a disjunctive formula $\varphi_1 \vee \varphi_2$ is considered true in those worlds for which *at least one of* φ_1 and φ_2 is true: that is, $\varphi_1 \vee \varphi_2$ is true in those worlds w for which $w \in \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!]$ *or* $w \in \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!]$. Thus, we define $\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \vee \varphi_2]\!]$ in terms of set union:

$$\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \vee \varphi_2]\!] = \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!] \cup \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!].$$

Implication: An implication $\varphi_1 \supset \varphi_2$ is true in those worlds w for which either φ_2 is true (i.e., $w \in \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!]$) or φ_1 is not true (i.e., $w \notin \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!]$, and thus $w \in \mathcal{E}_{\mathcal{M}}[\![\neg\varphi_1]\!]$). That is, $\varphi_1 \supset \varphi_2$ is true in those worlds in which, if φ_1 is true, then φ_2 is also true; if φ_1 is false, then φ_2's interpretation is immaterial. Thus, we define the semantics of implications as follows:

$$\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \supset \varphi_2]\!] = (W - \mathcal{E}_{\mathcal{M}}[\![\varphi_1]\!]) \cup \mathcal{E}_{\mathcal{M}}[\![\varphi_2]\!].$$

Equivalence: An equivalence $\varphi_1 \equiv \varphi_2$ is true in exactly those worlds w in which the implications $\varphi_1 \supset \varphi_2$ and $\varphi_2 \supset \varphi_1$ are *both* true. Thus, we define the semantics of implications as follows:

$$\mathcal{E}_{\mathcal{M}}[\![\varphi_1 \equiv \varphi_2]\!] = \mathcal{E}_{\mathcal{M}}[\![\varphi_1 \supset \varphi_2]\!] \cap \mathcal{E}_{\mathcal{M}}[\![\varphi_2 \supset \varphi_1]\!].$$

Example 2.12

Let $\mathcal{M}_1$ be the Kripke structure $\langle W_1, I_1, J_1 \rangle$ from Example 2.8. The set $\mathcal{E}_{\mathcal{M}_1}[\![q \supset (r \wedge s)]\!]$ of worlds in W_1 in which the formula $q \supset (r \wedge s)$ is true is calculated as follows:

$$\begin{aligned}
\mathcal{E}_{\mathcal{M}_1}[\![q \supset (r \wedge s)]\!] &= (W_1 - \mathcal{E}_{\mathcal{M}_1}[\![q]\!]) \cup \mathcal{E}_{\mathcal{M}_1}[\![r \wedge s]\!] \\
&= (W_1 - I_1(q)) \cup (\mathcal{E}_{\mathcal{M}_1}[\![r]\!] \cap \mathcal{E}_{\mathcal{M}_1}[\![s]\!]) \\
&= (W_1 - \{w_0, w_2\}) \cup (I_1(r) \cap I_1(s)) \\
&= \{w_1\} \cup (\{w_1\} \cap \{w_1, w_2\}) \\
&= \{w_1\} \cup \{w_1\} \\
&= \{w_1\}.
\end{aligned}$$ ◊

In the following example, we evaluate the same formula as in the previous example, but with respect to a different Kripke structure.

Example 2.13

Let $\mathcal{M}_2$ be the Kripke structure $\langle W_2, I_2, J_2 \rangle$ from Example 2.9. The set $\mathcal{E}_{\mathcal{M}_2}[\![q \supset (r \wedge s)]\!]$ of worlds W_2 in which the formula $q \supset (r \wedge s)$ is true is calculated as follows:

$$\begin{aligned} \mathcal{E}_{\mathcal{M}_2}[\![q \supset (r \wedge s)]\!] &= (W_2 - \mathcal{E}_{\mathcal{M}_2}[\![q]\!]) \cup \mathcal{E}_{\mathcal{M}_2}[\![r \wedge s]\!] \\ &= (W_2 - I_2(q)) \cup (\mathcal{E}_{\mathcal{M}_2}[\![r]\!] \cap \mathcal{E}_{\mathcal{M}_2}[\![s]\!]) \\ &= (W_2 - \{A, B, D\}) \cup (I_2(r) \cap I_2(s)) \\ &= \{C\} \cup (\emptyset \cap \{A, B, C, D\}) \\ &= \{C\} \cup \emptyset \\ &= \{C\}. \end{aligned}$$

$\Diamond$

Access-Control Operators

The access-control operators of the logic (e.g., says , controls , and $\Rightarrow$) have more interesting semantics.

Says: A formula P says φ is meant to denote a situation in which the principal P makes the statement φ. Intuitively, a principal should make statements that they *believe* to be true. What does it mean for a principal to believe a statement is true in a given world? The standard answer is that a principal P believes φ to be true in a specific world w if φ is true in all of the worlds w' that P *can conceive* the current world to be (i.e., all w' such that (w, w') is in $J(P)$). Of course, this set of *possible worlds* is simply the set $J(P)(w)$; φ is true in every world in $J(P)(w)$ if and only if $J(P)(w) \subseteq \mathcal{E}_{\mathcal{M}}[\![\varphi]\!]$. Therefore, we define

$$\mathcal{E}_{\mathcal{M}}[\![P \text{ says } \varphi]\!] = \{w \mid J(P)(w) \subseteq \mathcal{E}_{\mathcal{M}}[\![\varphi]\!]\}.$$

Controls: Formulas of the form P controls φ express a principal P's jurisdiction or authority regarding the statement φ. We interpret P controls φ as syntactic sugar for the statement $(P \text{ says } \varphi) \supset \varphi$, which captures the desired intuition: if the authority P says that φ is true, then φ is true. Thus, we give the meaning of P controls φ directly as the meaning of this rewriting:

$$\mathcal{E}_{\mathcal{M}}[\![P \text{ controls } \varphi]\!] = \mathcal{E}_{\mathcal{M}}[\![(P \text{ says } \varphi) \supset \varphi]\!].$$

Speaks For: To understand the semantics of formulas with form $P \Rightarrow Q$, recall the purpose of such formulas: we wish to express a proxy relationship between P and Q that will permit us to safely attribute P's statements to Q as well, independent of a particular world. That is, if $P \Rightarrow Q$, then it should be reasonable to interpret any statement from P as being a statement that Q would also make. In terms of the semantics, we have seen that a principal P making a statement φ in a world w means that $J(P)(w) \subseteq \mathcal{E}_{\mathcal{M}}[\![\varphi]\!]$. Thus, if we wish to associate *all* of P's statements to Q, then we need to know that $J(Q)(w) \subseteq J(P)(w)$ for all worlds w. If $J(Q) \subseteq J(P)$, then this relationship naturally holds. Therefore, we define

$$\mathcal{E}_{\mathcal{M}}[\![P \Rightarrow Q]\!] = \begin{cases} W, & \text{if } J(Q) \subseteq J(P) \\ \emptyset, & \text{otherwise.} \end{cases}$$

The following examples illustrate these semantic definitions.

Example 2.14

Recall $\mathcal{M}_0 = \langle W_0, I_0, J_0 \rangle$ from Example 2.7. The set of worlds in W_0 in which the formula *Hal* says *g* is true is given by $\mathcal{E}_{\mathcal{M}_0}[\![\mathit{Hal} \text{ says } g]\!]$, which is calculated as follows:

$$\begin{aligned} \mathcal{E}_{\mathcal{M}_0}[\![\mathit{Hal} \text{ says } g]\!] &= \{w \mid J_0(\mathit{Hal})(w) \subseteq \mathcal{E}_{\mathcal{M}_0}[\![g]\!]\} \\ &= \{w \mid J_0(\mathit{Hal})(w) \subseteq \{sw\}\} \\ &= \{sw, sc\}. \end{aligned}$$

This result captures *Hal*'s mistaken belief that, whenever it is sunny (i.e., when the current world is either *sw* or *sc*), the children will be able to go outside.

In contrast, recall that *Flo* is unable to distinguish the two worlds *sw* and *sc*. Specifically, the relation $J_0(\mathit{Flo})$ has the following properties:

$$\begin{aligned} J_0(\mathit{Flo})(sw) &= \{sw, sc\}, \\ J_0(\mathit{Flo})(sc) &= \{sw, sc\}, \\ J_0(\mathit{Flo})(ns) &= \{ns\}. \end{aligned}$$

Thus, the worlds in which *Flo* says *g* is true can be calculated as follows:

$$\begin{aligned} \mathcal{E}_{\mathcal{M}_0}[\![\mathit{Flo} \text{ says } g]\!] &= \{w \mid J_0(\mathit{Flo})(w) \subseteq \mathcal{E}_{\mathcal{M}_0}[\![g]\!]\} \\ &= \{w \mid J_0(\mathit{Flo})(w) \subseteq \{sw\}\} \\ &= \emptyset. \end{aligned}$$

That is, there are no worlds in which *Flo* is convinced that the children will be able to go outside. ◊

Example 2.15

Recall $\mathcal{M}_1 = \langle W_1, I_1, J_1 \rangle$ from Examples 2.8 and Example 2.12. The set of worlds in W_1 in which the formula *Alice* says $(q \supset (r \wedge s))$ is true is given by $\mathcal{E}_{\mathcal{M}_1}[\![\mathit{Alice} \text{ says } (q \supset (r \wedge s))]\!]$, calculated as follows:

$$\begin{aligned} \mathcal{E}_{\mathcal{M}_1}[\![\mathit{Alice} \text{ says } (q \supset (r \wedge s))]\!] &= \{w \mid J_1(\mathit{Alice})(w) \subseteq \mathcal{E}_{\mathcal{M}_1}[\![q \supset (r \wedge s)]\!]\} \\ &= \{w \mid J_1(\mathit{Alice})(w) \subseteq \{w_1\}\} \\ &= \{w_1\}. \end{aligned}$$

This result is not surprising, because *Alice* had perfect knowledge of the separate worlds: thus, she believes $q \supset (r \wedge s)$ to be true in precisely those worlds in which it is true. ◊