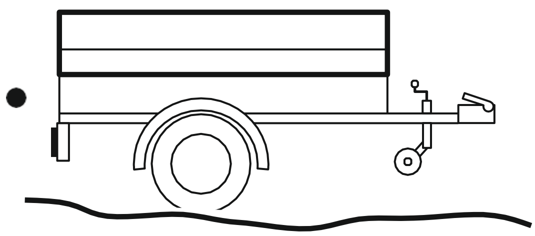
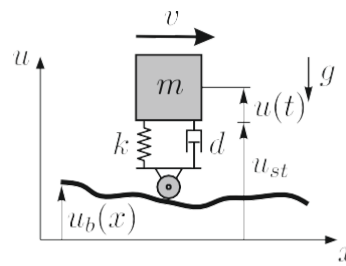
ASSIGNMENT 1

COMPUTATIONAL MODELLING

Task 1:





A uniaxial trailer (total mass m = 350 kg) moves with constant speed v = 1 *m/s* along an uneven road (road course u_b (x)) and starts to oscillate. Since we are only interested in the dipping motion of the trailer, a model with one degree of freedom, the 1-DoF system shown above, is used to study these oscillations. The equivalent *damping* d = 1000 Ns/m and equivalent stiffness k = 60000 N/m result from the design spring and damper elements and the wheels. The vertical displacement $u(t)$ of the hanger is always measured from the static rest position.Use MATLAB for programming subsequent parts along with numerical solutions.

a) Determine the equation of motion of the pendant for a general ground unevenness u_b (x).

---

b) Investigate the eigenbehavior of the pendant by first calculating the eigenvalues $\lambda_{1,2}$ of the equivalent system and plotting them in the complex number plane. What can you tell about the system behavior by the position of the eigenvalues alone? Then calculate the time history of the free oscillation in the real and complex notation and plot the displacement $u(t)$ and oscillation velocity $\dot{u}(t)$ of the pendant. Assume that the stationary trailer initially sinks by $u(0) = -0.01$ m due to a load and that the load is suddenly unloaded afterwards.

c) To illustrate the complex notation, represent the solution of the free oscillation additionally as a sum of two rotating pointers in the complex plane. What are the advantages of the complex notation?

d) The analytically calculated results are now to be compared with numerical calculations. Perform a transformation into the state space and apply different explicit one-step integration methods of different order to calculate the state of the system over time. How does the numerical solution behave as a function of the step size and the solver order? Contrast the analytical and numerical solution graphically.

---

e) To investigate the behavior of the trailer under an external load, the ground unevenness is first assumed to be a harmonic function u_b $(x) = 0.02 \cos(\Omega\, x)$. The excitation angular frequency can vary between 0 (static) and 5 times the natural angular frequency of the pendant ω_0 . Calculate the particulate solution of the equivalent system using the augmentation function $V(\eta, D)$ and the phase angle $\gamma(\eta, D)$. Plot the enlargement function V (η, D) and the phase angle $\gamma(\eta, D)$ for the D¨ampf-values d = [0, 600, 1000, 2000, 3000, 5000]. Where is the resonance point of the system and what is meant by it?

f) The stationary trailer is again loaded with a load which leads to a sinking of $u(0)$ = -0.01

m. The trailer is then jerked off and the load falls off the trailer. Afterwards the trailer starts to move jerkily and the load falls off the traeli.rIt can be assumed that the velocity v = 1 m/s is reached after a very short time $t \approx 0$. Calculate the time course of the hanger movement and its vertical velocity for different values of Ω and present the results graphically. What changes in the solution behavior if the excitation angular frequency Ω is very small, very large or approximately equal to the natural angular frequency ω_0 ?

g) Finally, the analytical results are to be compared again with numerical methods. Proceed analogously to point d).

---

h) For the investigations of the system behavior under periodic excitation, the beam unevenness is now represented by a periodic triangular function $u_b(x) = 0.02\,(|2x| - 1)$ for

$-1 < x < 1$ and $u_b(x + k - 2) = u_b(x)$ ($k \in Z$). First determine the time-varying load $p(t)$ acting on the pendant. Perform a Fourier series decomposition and calculate the first N = 50 elements of the Fourier series. Compare the time course of the load $p(t)$ and the Fourier series. How does the result change with decreasing and increasing number of elements N ?

i) Calculate the particle solution for the system using the truncated Fourier series, the augmentation function $V(\eta, D)$ and the phase function $\gamma(\eta, D)$. Furthermore, plot the amplitude and phase spectrum, as well as the time history, of the excitation and the system response. What conclusions can you already draw from the amplitude spectrum for the system response?

j) Determine the general system response for a scenario analogous to point f) and plot the results.

k) Finally, the analytical results are to be compared again with numerical methods. Proceed analogously to point d)

Task 2: A continuum body has the shape of a unit cube $\Omega_0 = (0, 1)^3$ (in the undeformed configuration). It undergoes the following motion

$$\chi(\mathbf{X}, t) : \begin{cases} x_1 = \lambda_1(t)X_1 + \gamma_1(t)X_2 + \alpha_1(t), \\ x_2 = \gamma_2(t)X_1 + \lambda_2(t)X_2 + \alpha_2(t), \\ x_3 = \lambda_3(t)X_3, \end{cases}$$

where $\gamma_i(t) \in \mathbb{R}$ and $\alpha_i(t) \in \mathbb{R}$ with $i = \{1, 2\}$ and $\lambda_j(t) \in \mathbb{R} > 0$ with $j = \{1, 2, 3\}$, and t denotes the time. In order to obtain $\mathbf{x}(\mathbf{X}, t = 0) = \mathbf{X}$ in the reference configuration $\Omega(t = 0) = \Omega_0$, $\gamma_i(t = 0) = \alpha_i(t = 0) = 0$, and $\lambda_j(t = 0) = 1$ needs to be fulfilled.

1. Sketch the reference and the current configuration of the continuum body, i.e. at $t = 0$ and $t = t_1$, respectively. Therefore, project the 3-dimensional body to the x_1–x_2 plane. Use the values $\lambda_1(t_1) = 1.6$, $\lambda_2(t_1) = 1.1$, $\lambda_3(t_1) = 1$, $\gamma_1(t_1) = 0.4$, $\gamma_2(t_1) = 0.2$, $\alpha_1(t_1) = 3$ and $\alpha_2(t_1) = 4$ only for this task. For all subsequent tasks, use the variables $\lambda_j(t)$, $\gamma_i(t)$ and $\alpha_i(t)$.

2. Which of the variables λ_j, γ_i and α_i are responsible for *translation*, *shearing* and *stretching* of the deformable body, respectively? Try to assign the terms by looking at $\chi(\mathbf{X}, t)$ for all variables separately.

3. Compute the displacement vector **u** from reference to current configuration and add the vector to your sketch from subtask 1 (use again the parameter values from subtask 1 to get numerical results).

4. Derive the deformation gradient **F**. Does **F** still include rigid body translation?

5. Calculate the volume ratio J and describe the meaning of it with respect to the current and reference volume. Which assumptions can be made for an *incompressible solid material* with respect to J?

6. Express the right Cauchy-Green tensor **C** in terms of the right (pure) stretch tensor $\mathbf{U} = \mathbf{U}^T$ by using the multiplicative split (polar decomposition) of the deformation gradient **F**, i.e. $\mathbf{F} = \mathbf{RU}$, where **R** is a proper orthogonal (pure) rotation tensor ($\mathbf{R}^{-1} = \mathbf{R}^T$). Why is **C** preferably used for constitutive models compared to the deformation gradient **F**? Compute **C** for the given motion $\chi(\mathbf{X}, t)$.

7. Derive the left Cauchy-Green tensor **b**.

8. Compute **E** and **C** for **F** = **I**. What is the difference between **E** and **C**?

9. Given $\lambda_1(t) = e^{8t}$, $\lambda_2(t) = e^{-13t}$, $\lambda_3(t) = e^{t}$, $\gamma_1(t) = t^4$, $\gamma_2(t) = t^2$ and $\alpha_1(t) = \alpha_2(t) = 5t$. Show that the material velocity gradient $\mathrm{Grad}\mathbf{V}(\mathbf{X}, t)$ is equal to the material time derivative of the deformation gradient $\dot{\mathbf{F}}$. Use this specific time dependency of the variables λ_j, γ_i and α_i only for this task.

Task 3: We would like to model the mechanical behavior of a thin incompressible rubber-like membrane under biaxial deformation. Assuming this material as *incompressible*, *homogeneous*, *isotropic* and *hyperelastic*. Therefore we are using the so-called neo-Hookean model, defined in terms of the first principal invariant I_1, i.e.

$$\Psi = \frac{\mu}{2}(I_1 - 3) - p(J - 1), \tag{1}$$

where the parameter μ denotes the shear modulus and p is the Lagrange multiplier enforcing incompressibility. Consider the motion $\boldsymbol{\chi}(\mathbf{X}, t)$, as given in Problem 1, and set $\alpha_1 = \alpha_2 = 0$ and $\gamma_1 = \gamma_2 = 0$ for $t \geq 0$, which corresponds to pure biaxial extension without a shear deformation.

In general, the Cauchy stress tensor $\boldsymbol{\sigma}$ for isotropic materials (defined in terms of the principal invariants) can be calculated as

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\frac{\partial \Psi}{\partial I_1}\mathbf{b} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{b}^{-1}. \tag{2}$$

For all subsequent tasks, generate the requested plots using software packages like Matlab, Python, etc.

1. Figure 1 depicts an infinitesimally small cube (cut from the membrane) with arrows rep- resenting the nine Cauchy stress tensor components. Assign the individual components σ_{ij} to the arrows. Note that the first index i refers to the normal direction $\mathbf{n}_i$ of the surface and the second index j to the effective direction of the stress. Which of these components are referred to as *normal stresses* and which as *shear stresses*?

2. Calculate the matrix representation of the Cauchy stress tensor $\boldsymbol{\sigma}$. In order to eliminate the unknown Lagrange multiplier p, use the thin membrane theory (i.e. $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$). Express λ_3 in terms of λ_1 and λ_2 using the incompressibility condition.

3. Now consider an equi-biaxial loading case, i.e. $\lambda_1 = \lambda_2 \in [1, 3]$. For $\mu = 1$ MPa plot the principal Cauchy stress σ_1 over the principal stretch λ_1. Furthermore, plot the functions $f_1(\lambda_1) = \mu\lambda^2$, and $f_2(\lambda_1, \lambda_2) = \mu(\lambda_1\lambda_2)^{-2}$ into the same figure and discuss how they contribute to σ_1, e.g., which function is dominant at low and high stretch levels, respectively. Do not forget to label the axes and use legends to identify the curves.

4. Next, we would like to study the effect of the material parameter μ onto the stress-stretch behavior. For $\mu = \{0.5, 1.0, 2.0\}$ MPa plot the principal Cauchy stresses σ_1 and σ_2 over the principal stretches λ_1 and λ_2, respectively. Describe the influence of the

changes of μ onto the stress-stretch behavior and by means of your plots discuss why the neo-Hookean model can be used to describe isotropic materials (what does isotropic material behavior mean?)

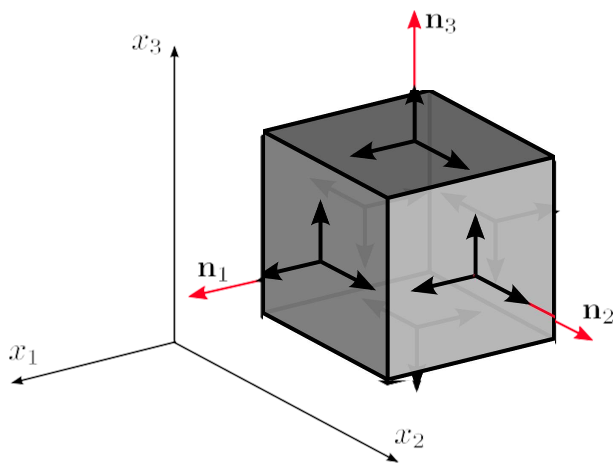


Figure 1: Components of $\boldsymbol{\sigma}$ that completely define the state of stress at a point inside a material.

5. Compute the matrix representation of the first Piola-Kirchhoff stress tensor **P** from $\boldsymbol{\sigma}$ via the Piola transformation.
6. Discuss the general difference between the Cauchy stress $\boldsymbol{\sigma}$ and the first Piola-Kirchhoff stress **P**. In which configurations do the first Piola-Kirchhoff and the Cauchy stress ten- sors live? Furthermore, explain why for materials like steel the consideration of the first Piola-Kirchhoff stress (also known as engineering stress) is sufficient in typical mechanical engineering applications but for soft biological tissues the Cauchy stress needs to be considered. For better argumentation plot σ_1 and P_1 over the principal stretch $\lambda_1 \in [1, 2]$ into the same figure.