MA 2002D Mathematics IV
Winter Semester 2022-2023
Tutorial sheet I (Common to All Branches)

1. Prove that $f(z)=|z|^2$ is continuous everywhere but differentiable nowhere except at origin.
2. If a function *f (z)* is analytic, show that it is independent of $\bar{z}$.
3. If the analytic function *f(z) = u + iv* is expressed in terms of polar co-ordinates, show that $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r}=\frac{-1}{r}\frac{\partial u}{\partial \theta}$. Also, show that its real and imaginary parts are solutions of Laplace equation in polar co-ordinates given by $\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}=0$.
4. If *f(z)* is analytic prove the following.

   (i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ \text{In}\ |f'(z)|=0$ (ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$
5. Determine whether the following functions are analytic or not. If analytic find its derivative
   - (i) f(z) = $(z^2-2)\ e^{-x}$ (cosy – i siny)
   - (ii) f(z) = log z
   - (iii) f(z) = cosx.coshy – i sinx. sinhy
   - (iv) f(z) = sinh z
   - (v) f(z) = e^{3z}
   - (vi) f(z) = cos z
   - (viii) f(z) = $z\bar{z}$
6. Given the following functions, show that C-R equations are not sufficient for differentiability at the point specified.

   (a) $f(z)=\sqrt{|xy|}$ at z = 0 (b) $f(z)=\begin{cases}\frac{xy}{x^2+y^2} & z\neq 0\\ 0 & z=0\end{cases}$ at z = 0.

   (c) $f(z)=\begin{cases}\frac{z^5}{|z|^4}, for\ z\neq 0\\ 0,\ for\ z=0\end{cases}$ at z = 0.

7. Check for analyticity of the following functions

   (a) $\frac{i}{z^5}$ (b) Re(z^3) (c) $\frac{\text{Re}(z)}{\text{Im}(z)}$

   (d) f(z) = cosx . coshy + i sinx . sinhy (e) f(z) = z - $\bar{z}$

   (f) f(z) = $2x+ixy^2$ (g) f(z) = $|z|^2$
8. If f(z) and $\overline{f(z)}$ are analytic in a region D, show that f(z) is constant in that region.
9. Prove that an analytic function whose real part is constant is a constant function
10. Determine the constants a and b such that the function, $f(z)=(x^2+ay^2-2xy)+i(bx^2-y^2+2xy)$ is analytic. Also, find its derivative.
11. If f(z) = u + iv is analytic in a region D and v = u^2 in D, then f is a constant.
12. If f(z) is analytic in D, then f(z) is a constant if

    (i) $|f(z)|$ is constant (ii) $f'(z)=0$
13. Show that if a function f(z) = u + iv analytic in a domain R and if u and v have continuous second order partial derivatives, then u and v satisfy the Laplace Equation. i.e. $\nabla^2 u=0$ and $\nabla^2 v=0$.

14. Find an analytic function whose imaginary part is $3x^2y - y^3$ and which vanishes at z = 0.
15. Check whether the function sinz and cosz satisfies the following properties:
    (i) $f(z+2\pi) = f(z)$; (ii) $|f(z)| \le 1$.
16. Check whether f(x + iy) = $\frac{1}{2}$ log ($x^2 + y^2$) + i $\tan^{-1}$(y/x) ; ($x^2 + y^2 \neq 0$) is analytic. If so , find $f'(z)$.
17. Determine whether the following functions are harmonic. If so, find the corresponding analytic function f(z) = u + iv
    (a) $u = \dfrac{xy}{x^2+y^2}$ (b) u = e^{2x}(xcos2y) (c) v = $(x^2 - y^2)^2$
    (d) v = $-e^{-x}$ siny
    (e) u = sinx coshy + 2cosx sinhy + $x^2 - y^2$ + 4xy
    (f) v = e^{-x} [2xy cosy + ($y^2 - x^2$) siny]
    (g) u = sinx. coshy

18. Find the analytic function f(z) = u + iv if
    (a) $u - v = (x-y)(x^2+2xy+y^2)$ (b) $u + v = \dfrac{x}{x^2+y^2}$, $f(1) = 1$
    (c) $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos 2x}$ and $f(\frac{\pi}{2}) = \frac{3-i}{2}$
    (d) $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$
19. Prove that u(x, y) = $x^2 - y^2$ and v(x, y) = $\dfrac{-y}{x^2+y^2}$ are both harmonic but u + iv is not analytic.
20. If f(z) is analytic show that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$
21. Find all solutions of the following equations i) $e^z = 2i$ ii) $\sin z = i$.
22. Prove that there cannot exist an analytic function on a region with real part $x - 2y^2$.
23. If $f(z)$ is an analytic function in a region, then show that $f'(z)$ is also analytic in that region.
24. Find the image of the square region with vectors (0,0), (1,0), (1,1) and (0,1) under the transformation w = 2z - i .
25. Find the image of the rectangular region bounded by x = 0, y = 0, x = 2, y = 1 under the (i) translation w = z + (1-2i) (ii) rotation w = iz (iii) transformation w = (1 + i)z + (2 - i).
26. Find the image of the region y > 1 under the transformation w = iz + 1.
27. Find the image of the semi infinite strip x > 0; 0 < y < 2 under the transformation w = iz + 1.
28. Show that by means of the inversion w = $\frac{1}{z}$, the circle given by $|z - 2| = 7$ is mapped into the circle $|w + \frac{2}{45}| = \frac{7}{45}$.
29. Find the image of the triangle with vertices i, 1+ i, 1-i in the z – plane under the transformation w = 3z + 4 – 2i .
30. Find the image of the following regions under w = $\frac{1}{z}$ (i) the strip $0 < y < \frac{1}{2}$, (ii) the circle, $|z - 3i| = 3$.
31. Show that w = $\dfrac{z-1}{z+1}$ maps the half plane $x \ge 0$ on to the unit circle $|w| \le 1$. Show also that this transformation maps the half plane $y \ge 0$ on to the half plane $v \ge 0$.

32. Find the region in the w – plane in to which the region $\frac{1}{2} \le y \le 1$ is mapped by the transformation $w = z^2$.

33. Under $w = \frac{1}{z}$, find the image of (i) $|z - 2i| = 2$ (ii) $\frac{1}{4} \le y \le \frac{1}{2}$. Also show the regions graphically.

34. Show that $w = \frac{2z+3}{z-4}$ maps $x^2 + y^2 - 4x = 0$ on to $4u + 3 = 0$.

35. Find regions where the following mappings are conformal and also find their critical points.
(i) $w = z^3$ (ii) $w = \cos z$ (iii) $w = \sinh z$.

36. Show that $w = z^2$ maps the circle $|z - 1| = 1$ in to the cardioids $\rho = 2(1 + \cos\phi)$ where $w = \rho e^{i\phi}$ in the w – plane.

37. Determine the region of w – plane in to which the first quadrant of z – plane is mapped under the transformation $w = z^2$.

38. Discuss $w = e^z$ and show that it transforms the region between $y = 0$ and $y = \pi$ in to the upper half of w – plane.

39. Show that $w = \frac{z-i}{z+i}$ maps real axis in z – plane in to $|w| = 1$. What portion of the z – plane corresponds to the interior of the circle in the w – plane.

40. Find the images of $x = 0$, $x = 1$, $y = 0$ and $y = 1$ under $w = z^2$.

41. Discuss the transformation $w = \cos hz$ and find the image of the semi-infinite strip $x \ge 0$, $0 \le y \le \pi$ of z – plane.

42. Find the image of the region $0 < x < 2\pi$, $1 < y < 2$ under $w = \sin z$.

43. Find the fixed points of the transformations $w = \frac{3z+2}{z-1}$.

44. Prove that $w = \frac{iz+1}{z+i}$ maps the part of the real axis between $z = 1$ and $z = -1$ as a semi-circle in the w –plane.

45. Find the bilinear transformation which maps

i.	$z = 1, -i, -1$	in to	$w = 2, 0, -2$	respectively
ii.	$z = 1, i, -1$	in to	$w = 0, 1, \alpha$	
iii.	$z = -1, 1, \alpha$	in to	$w = -i, -1, i$	
iv.	$z = 0, 1, i$	in to	$w = \frac{-1}{2}, 0, -1+i$	
v.	$z = 0, -1, \alpha$	in to	$w = -1, -2-i, i$	
vi.	$z = 1+i, -i, 2-i$	in to	$w = 0, 1, i$	
vii.	$z = \infty, 1, -1$	in to	$w = 1, \frac{3+2i}{5}, 3-2i$	
viii.	$z = 2, i, -2$	in to	$w = 1, i, -1$	
ix.	$z = \alpha, 0, -1$	in to	$w = 1, 0, \frac{1+i}{2}$	
