Wait Time Information Design

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When customers arrive, service providers often collect information to generate delay forecasts. We study how delay data-collection and forecasting systems can be designed to improve customer satisfaction. We assume that customers may be loss-averse in the sense that an increase in the expected wait causes more distress than the positive response caused by an equivalent decrease and that they may be risk conscious in that an increase in the variance of expected delay reduces utility. Our goal is to find the structure of delay information that optimizes the customers' experience while waiting. Delay forecasts follow Bayes' rule, given a prior distribution, the additional information collected for a particular customer, and the passage of time. We find that when loss aversion dominates, the optimal delay information focuses on the tails of the delay distribution. When risk consciousness is dominant more traditional information about the duration of delay– along a continuum from 'short' to 'long'–is optimal, and this information should be most precise about the longest delays. The optimal information design also affects the timing of delay revelation. When customers are loss averse, it is optimal to avoid changes in expected delay over time, so that waiting times are revealed as customers go into service. When customers are risk conscious, it is optimal to provide information so that they learn the good (or bad) news immediately, when they arrive.

Key words: delay information, loss aversion, customer satisfaction, forecasting

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"***People do not remember what you say, they remember how you made them feel***"[1]

1. Introduction

When a customer has to wait, service providers often share delay forecasts that are derived from data specific to that customer. In an emergency department (ED), for example, a delay forecast may be based on a particular patient's Emergency Severity Index (ESI) and the level of congestion in the ED. It is usually assumed that service providers should collect this information, and formulate forecasting models, to minimize the mean-squared error (MSE) or similar measures of forecast inaccuracy (e.g., Ang et al. 2016; Bassamboo and Ibrahim 2021; Kuo et al. 2020).

[1] Source uncertain. This saying, and variations, have been attributed to Maya Angelou and Carl W. Buehner

The impact of a forecast on a customer's experience, however, may be influenced by factors besides accuracy at one point in time. Therefore, in this paper we generate forecasts to optimize each customer's experience over the duration of the delay before service. Our model of the customer's experience incorporates two potential sources of disutility. The first source stems from the asymmetric impact of forecast inaccuracy: an under-estimate in the expected wait may cause more distress than the positive response caused by an equivalent over-estimate (Yu 2020). Following Kahneman and Tversky (1979), we use the term *loss averse* to describe customers who experience greater disappointment when delays are longer than expected.

The second source of disutility accounts for the anxiety caused by uncertainty; as Maister (1985) wrote, "Uncertain waits are longer than known, finite waits." The traditional MSE is one measure of forecast quality, but the MSE is only calculated once, to compare the forecast with the realized delay. Our measure of the disutility caused by uncertainty, on the other hand, is accumulated over the duration of the delay as customers receive forecast information from the service provider, as time passes, and as customers update their delay expectations. We use the term *risk conscious* (Anunrojwong et al. 2020) to describe customers who are sensitive to this uncertainty throughout their wait. Given an objective function that takes loss aversion and/or risk consciousness into account, our model allows us to identify the type delay information (e.g., about customers and congestion) and the resulting forecast that is most valuable for maximizing customer satisfaction.

Our model assumes that before arrival, all customers share a common prior delay distribution that is known to both the service provider and the customers. When a customer arrives the service provider collects customer-specific data that provides imperfect information about customer's actual waiting time. Then the service provider uses Bayes' rule to update the customer's delay distribution given the information and communicates this posterior distribution to the customer. We call this shared posterior distribution the *delay forecast*. Our focus is on the design of the personalized information to be collected for each customer and therefore the optimal attributes of the forecast. For example, for an ED, our model shows that it may be worthwhile to put more effort into collecting detailed diagnostic and congestion information to improve the precision of forecasts for patients experiencing the longest, rather than intermediate, delays.

Our approach to this information design problem is general; in our initial framework we make no assumptions about the shape of the prior distribution and the structure of the delay information collected by the service provider. In traditional forecasting systems, data are usually used to classify patients along a continuum of the duration of delay, e.g., 'short', 'medium', and 'long.' We call this an *ordinal* information structure. Our model also allows for any other type of information structure, such as the identification of the very longest delays, or the classification of patients into rough categories such as 'extreme' (longest or shortest delays) vs. close to the middle.This allows us to identify the globally optimal structure that may be distinct from what is typically seen in practice, and therefore encourage practitioners to identify new sources of data to improve the customer experience.

In general, poor design of delay forecasts can cause direct harm, as customers suffer from emotional whiplash and make plans based on poor information. Poor forecast design can also lead to other harmful behaviors, such as emergency department patients who give up and leave without being seen by a provider. Our model of customer utility, with both loss aversion and risk consciousness, may be seen as an antecedent to such reneging behavior, although here we will assume that customers do not abandon while waiting. Therefore, the model directly applies to those ED's in which nearly all patients who require service do not leave without being seen (e.g., under 2% of patients leave without being seen at the Level 2 trauma center studied in Polevoi et al. 2005). It also applies to situations when a customer is committed to waiting, e.g., for the completion of a service that has started and cannot be interrupted such as auto, cell phone and computer repair.

Analysis of our model and our numerical experiments show that when customers are loss averse, an ordinal information structure is not always optimal. This is because such structures may lead to forecasts that generate unwarranted hope of a short delay, followed by disappointment while waiting. We identify an information design that performs better: data that excludes customers that are in either tail of the delay distribution. This information minimizes variation in expected delays over time, and hence minimizes customer disappointment as time passes. However, forecasts generated from this information leaves tail customers initially with a great deal of uncertainty–they do not know whether they are in the left or right tail–and hence is not well suited for risk-conscious customers. For risk-conscious customers, we find that ordinal designs are nearly always optimal, and information about the longest delays should leave the least residual uncertainty (the highest precision), with precision decreasing for shorter delays. We find, as well, that if we constrain information to be ordinal, then for loss averse customers it is optimal to be the most precise about *both* the shortest and longest delays. In general, when designing information and forecasts to maximize utility, our model highlights trade-offs among (i) the initial impact of a delay forecast, (ii) changes in utility as time passes, and (iii) the final effect on utility when the customer enters service.

The next section reviews related literature on delay information and customer satisfaction. Section 3 describes the model and works through a representative numerical example. In Section 4 we analyze the model when there is a single delay forecast at the time the customer first arrives, as is common in practice. The Section focuses on tractable special cases: three possible delay outcomes in §4.1 and a continuous delay distribution with loss-averse customers in §4.2. Section 5 describes numerical experiments motivated by the analytical results from Section 4; the experiments cover a wide range of delay distributions and explores optimal information strategies for customers who may be loss averse, risk conscious, or a mixture of both. While Sections 4-5 focused on a single delay forecast, Section 6 considers multiple delay forecasts, broadcast throughout the customer's waiting time. Finally, Section 7 lays out a set of basic principles for optimal delay information design based on the previous analytical results and numerical experiments, and the section concludes with a discussion of the practical implications of the work and topics for further research.

2. Literature Review

Within the economics literature our model has much in common with Ely et al. (2015), who describe how to design entertainment experiences, such as a mystery novel or a sporting event, with an optimal information revelation sequence so as to maximize the consumer's pleasure from suspense and surprise. Our model, however, is formulated to describe the experience of waiting, and a crucial aspect of our model is that information is revealed by the passage of time as well as by information collected and shared by the service provider.

Guda et al. (2020) use the framework developed by Ely et al. (2015) to analyze process transparency. They consider whether to provide information about progression through a multi-staged task to delay-sensitive consumers. They model the evolution of the beliefs of loss-averse consumers about the completion of the whole task over time and study under which circumstances revealing the completion of subtasks improves the consumer experience. They find that loss aversion favors not sharing progress information, while diminishing sensitivity (to news) does the opposite. In their paper, the firm's decision is binary (whether to reveal the completion of a subtask or not), while in ours, we identify among all possible information structures those that are consistent with Bayes' rule and minimizes disutility from both loss aversion and sensitivity to uncertainty.

Also related is the literature on Baysian persuasion, as introduced by Kamenica and Gentzkow (2011). They describe an information design problem in which a 'Sender' communicates signals to a 'Receiver' about an uncertain state of the world. Upon receiving a signal, the Receiver chooses an action, and payoffs to the Sender and Receiver depend on both the action and the state. The Sender's design problem is to choose signals that maximize her payoff. This framework has been applied to numerous settings, including but not limited to: Internet advertising (Rayo and Segal 2010), communication in organizations (Jehiel 2014), government (Gehlbach and Sonin 2014), medical research (Kolotilin 2015, Schweizer and Szech 2018), and banking (Gick and Pausch 2012, Goldstein and Leitner 2018). While the models in these papers assume that the agents who receive messages are expected-utility-maximizers, Anunrojwong et al. (2020) develop a convex optimization framework for the persuasion problem when the Receiver's utility is nonlinear in beliefs about the uncertain state. Their model allows for a wide range of utility functions for Receivers who are sensitive to uncertainty, and they call such Receivers *risk conscious*. For our model, we will use this term for customers who are sensitive to uncertainty in delay.

Note that in the Bayesian persuasion framework, the Sender has a utility function that differs from the Receiver, and the Sender designs signals to influence the Receiver's actions. In our model the Receiver does not choose an action. Rather, the Sender's goal is to maximize the utility of the Receiver; their utilities are aligned.

Communicating delay information is important in service operations because it may affect both customers' evaluation of service (Hassin 1986) and system throughput (Hassin 2007). Most analytical research in this area models customers as using delay information to estimate the distribution of delay, to determine expected waiting costs, and then to decide whether to wait in queue or leave. In the call center setting, Whitt (1999) analytically shows that if the service provider announces anticipated delays, customers are more likely to balk (leave immediately upon arrival) when all servers are busy than renege (leave after waiting for some time) and this reduces the system's average delay. Following up on Whitt (1999) researchers have also considered different levels of information (Guo and Zipkin 2007), more general forms of balking and abandonment (Armony et al. 2009), the credibility of announcements (Allon et al. 2011), priority queueing systems (Yu et al. 2017), and heterogeneity in the information available to customers (Hu et al. 2017).

Allon and Bassamboo (2011) focus on the timing of the provision of delay information and Cui and Veeraraghavan (2016) examine the impact of information about the service parameters of the system, rather than delay information. Lingenbrink and Iyer (2019) apply the Bayesian persuasion framework to a queueing system where customers may balk or abandon, and Anunrojwong et al. (2022) apply the framework to queueing systems with users who are heterogeneous in their ability to access alternative service options. Recall from above that Anunrojwong et al. (2020) also apply Bayesian persuasion to agents with nonlinear utility structures; a special case of their model is the problem of optimally sharing waiting time information with risk-conscious customers. Shumsky (1998) examines how forecasts of airline delays can be optimized to improve air traffic management, given a penalty when forecast errors accumulate over time (analogous to risk-conscious customers) as well as a cost to update forecasts.

There is also a growing empirical literature that documents the impact of delay announcements on customer behavior and satisfaction. Hui and Tse (1996) study how customers evaluate services given delay announcements, while Akşin et al. (2016), focuses on customer abandonment. Yu et al. (2016) use call-center data to estimate the true cost of waiting. Yu et al. (2021) conduct a field experiment to measure the degree of customer loss aversion and the impact of delay announcements. This study formulates a structural model in which customers update their delay expectations, given the service provider's announcement. This is similar to the Bayesian updating incorporated into our model, although we overlay a structure that allows the service provider to determine the optimal information structure. Yu et al. (2020) conduct a field experiment with a ride-sharing platform to assess the impact of the magnitude and frequency of delay announcements on abandonment from a virtual queue. Mogre et al. (2020) conduct an experiment to show how lead-time quotes (or delay announcements) serve as reference points for customer beliefs, and based on their results they formulate a model in which the firm chooses a price and lead-time quote to maximize total welfare in a single-server queue.

In most of these papers–both analytical and empirical–reneging and balking are used as proxies for the customer's experience, as customers anticipating a long delay are more likely to quit waiting or not join the queue at all. In this paper, we formulate a model for how delay forecasts affect the customer's experience while they wait. That is, our model seeks to capture the emotional impact of delay information on the entire waiting experience, from arrival until entering service.

There is also a stream of papers in the ED setting to study the effects of publishing delay information on patients' hospital selection (Xie and Youash 2011), coordination within hospital networks (Dong et al. 2015) and patient satisfaction (Shah et al. 2015, Ansari et al. 2018). Both Shah et al. (2015) and Ansari et al. (2018) study the impact of announcing delays on patient satisfaction. Shah et al. (2015) observe that the patients who received delay information were more satisfied overall. Ansari et al. (2018) find that announcements that overestimate delays lead to an increase in patient satisfaction due to patients' loss aversion, but announcements that overestimate delays by too much may have negative impacts on patients' waiting experience. While these two papers empirically document the relationship between patient satisfaction and delay announcements, they do not address the problem of how to design optimal forecasts.

In sum, the existing literature has primarily focused on the *expected wait*, and the decision of whether to join or leave the system given that expectation, while the focus of this paper is on modeling and optimizing customers' satisfaction with regard to their *experienced wait.*

3. A Model of Delay Information and Customer Satisfaction

Consider a customer who arrives at time $t = 0$ and whose service starts at some time $\omega \in \{1, 2, \ldots, \bar{\omega}\}$, where $\bar{\omega}$ is the latest possible service start-time. Note that we will use the terms 'delay' and 'service start time' interchangeably. That is, a delay of duration ω implies that service begins at time ω. For now, assume that both the underlying model of time t and the delays themselves are discrete; we will consider a continuous-time model in Section 4.2.

We will use subscripts to represent the service start-time for a particular customer and superscripts to represent the current time as time moves forward. The service start-time is distributed as a random variable Ω with probabilities $g_\omega^{-1} = \Pr(\Omega = \omega)$, $\omega \in \{1, 2, \ldots, \bar{\omega}\}$, mean $E[\Omega] = v^{-1}$, and distribution $\boldsymbol{g}^{-1} = (g_1^{-1}, g_2^{-1}, ..., g_{\bar{\omega}}^{-1})$. The superscript -1 indicates that this distribution is a *prior* distribution and is known to both the service provider and the customer before the customer arrives. In an ED, for example, the prior distribution may depend on the day of week, the time of day, and other factors known to both the patient and service provider. In addition, the customers' (accurate) priors may be formed from delay information that is available through the Internet (e.g., Groeger (2019), Health (2012)), billboards showing real-time waiting-times, word of month, and past experience.

For each customer arrival, the service provider uses an information system to generate specific data about that customer, e.g., customer attributes and real-time congestion data. The service provider uses Bayes' Rule to incorporate the new information into the prior, generating a posterior delay distribution that is then shared with the customer.[2] We use the term *forecast* to indicate the posterior distribution generated by the service provider and *announcement* to indicate that forecasts are shared with all customers in a particular period. We assume that the number and timing of announcements is exogenous and will focus on two cases: a single announcement when customers arrive (this Section and the following two Sections), and multiple announcements in each and every period that the customer waits (Section 6). We also assume that the service provider is truthful when reporting the forecast.

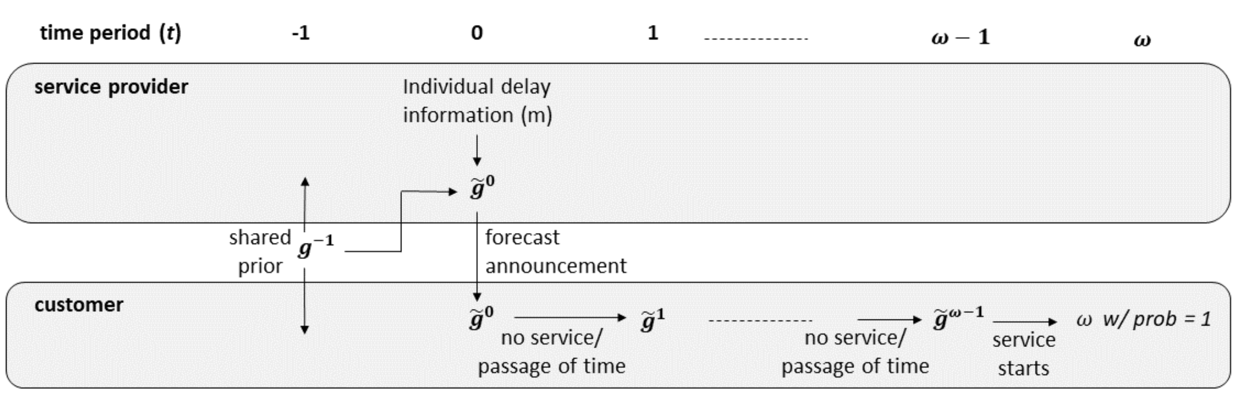


Figure 1 **Timeline of updates of delay distribution $\tilde{\boldsymbol{g}}^t$ for a customer with delay ω and a single announcement at time $t = 0$.**

After the forecast announcement at $t = 0$, the customer continues to update the delay distribution, given the passage of time, but without additional information from the service provider. Figure 1 shows the information m available to the service provider as well as the evolution over time of the distribution $\tilde{\boldsymbol{g}}^t$. Section 3.1 will describe the service provider's information structure and how

[2] For example, delay distributions may be communicated via a visualization of the distribution or a heat map.

the service provider and customers update distributions, Section 3.2 will describe the customer's utility and the service provider's objective, and Section 3.3 will step through a numerical example to show how the customer experience is shaped by a forecast derived from an ordinal information structure.

3.1. Delay Information and Forecasts

Consider a customer who arrives at $t = 0$ and will enter service at $t = \omega$. When the customer arrives, the service provider's system generates information $m \in \mathcal{M} = \{A, B, C, \cdots\}$; we call realization m the information system's *outcome*. Let $f_\omega(m)$ be the probability that outcome m is observed when the actual start time is ω, $0 \leq f_\omega(m) \leq 1$ and $\sum_{m \in \mathcal{M}} f_\omega(m) = 1$. The cardinality of $\mathcal{M}$ is M; this is the number of distinct outcomes, or delay categories, identified by the information system. For example suppose that $\bar{\omega} = 6$, $M = 2$, $f_\omega(A) = 1$ for $1 \leq \omega \leq 3$ and $f_\omega(B) = 1$ for $4 \leq \omega \leq 6$. Therefore, this information system would be capable of distinguishing between the three shortest and three longest delays; if each index t represents a half hour of delay, then this information system can determine whether a particular customer's delay is less than, or greater than, 90 minutes. For a given cardinality M, we will optimize the model over the values of $f_\omega(m)$ so that the solution will characterize the requirements of the information system that maximizes customer utility.

This model provides us with two levers to describe the amount of information provided by the data-collection system. First, the cardinality M describes the system's granularity. When $M = \bar{\omega}$, the technology is fine enough to distinguish among all service start-times, while $M = 1$ implies that the technology is not capable of partitioning the set of possible start-times and is therefore useless. In general, a larger M implies a more sophisticated technology.

Second, the range of $f_\omega(m)$ captures the discriminatory power of the system. We will assume that $f_\omega(m) \in \{0, 1\}$. This implies a high degree of discrimination–that is, the system can distinguish perfectly between 'adjacent' delays. This assumption helps to make the problem more tractable and leads to more intuitive results. Given that $f_\omega(m) \in \{0, 1\}$, an information structure corresponds to a partition of $\{1, ..., \bar{\omega}\}$ into M subsets, and optimization allows us to see which delays should be grouped into distinct delay categories. In the concluding Section 7 we will discuss information systems where $0 < f < 1$.

Via Bayes' rule, the updated delay distribution in period $t = 0$ after observing outcome m is

$$g_T^0(m) = \frac{g_t^{-1} f_T(m)}{\sum_{\tau=1}^{\bar{\omega}} g_\tau^{-1} f_\tau(m)}, \text{ for } 1 \leq T \leq \bar{\omega} \text{ and } m \in \mathcal{M}.$$

The service provider shares this distribution with the customer as an updated delay forecast at the end of $t = 0$. Then, as the customer waits, the service provider does not make additional announcements. The passage of time, however, enables the customer to update the delay distribution. In period t, the updated distribution is:

$$g_T^t(m) = \frac{g_T^{t-1}(m)}{\sum_{\tau=t+1}^{\bar{\omega}} g_\tau^{t-1}(m)}, \text{ for } 1 \le t \le \bar{\omega}-1,\ t+1 \le T \le \bar{\omega} \text{ and } m \in \mathcal{M}.$$

We use $\tilde{x}$ to indicate that x is a random variable and bold $\boldsymbol{x}$ to indicate that that $\boldsymbol{x} = (x_1, x_2, \ldots)$ is a vector. As the outcome m depends on the actual, realized start time, the updated distribution is a random variable. With this notation, let the customer's updated delay distribution in period t be denoted by $\tilde{\boldsymbol{g}}^t = (\tilde{g}_{t+1}^t, \tilde{g}_{t+2}^t, ..., \tilde{g}_{\bar{\omega}}^t)$. Let the realized *sequence* of updates for a customer who receives his service at the beginning of period ω be $\tilde{\boldsymbol{g}}_\omega = (\boldsymbol{g}^{-1}, \tilde{\boldsymbol{g}}^0, \tilde{\boldsymbol{g}}^1, ..., \tilde{\boldsymbol{g}}^{\omega-1})$. The utility of the customer who will be served at time ω, $U_\omega(\tilde{\boldsymbol{g}}_\omega)$, depends on the sequence of updates until $\omega - 1$, $\tilde{\boldsymbol{g}}_\omega$, and the start time ω (when a customer goes into service, the distribution collapses to ω with probability = 1.). We will describe the components of the customer's utility below.

Let $\boldsymbol{f} = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{\bar{\omega}})$, where $\boldsymbol{f}_\omega = (f_\omega(A), f_\omega(B), \ldots)$ be the information outcome probabilities for a customer who receives service in period ω. From the Bayesian updating equations above, it is clear that the sequence of updates $\tilde{\boldsymbol{g}}_\omega$ depends on $\boldsymbol{f}$. Now, we can define the service provider's optimization problem:

$$\max_{\boldsymbol{f}} \ \mathbb{E}_\Omega\{U_\Omega(\tilde{\boldsymbol{g}}_\Omega(\boldsymbol{f}))\} \tag{1}$$

Given that $f_\omega(m) \in \{0, 1\}$, a solution partitions $\{1, ..., \bar{\omega}\}$ into M subsets, each corresponding to a unique outcome. It is important to note that the meaning of each outcome only becomes clear when the partitioning is given. As the goal will be to find the partitioning that maximizes expected customer utility, the meaning of each outcome is endogenous to our model. Consider again the example with $\bar{\omega} = 6$ and with $f_\omega(A) = 1$ for $1 \le \omega \le 3$ and $f_\omega(B) = 1$ for $4 \le \omega \le 6$. Therefore delays are partitioned with $\omega \in \{1, 2, 3\}$ generating outcome A and $\omega \in \{4, 5, 6\}$ generating outcome B; hence, outcome A (B) means 'shortest' ('longest') delays. Using alternate notation, this partition may simply be described as $AAABBB$. We say that this information design has an *ordinal* structure, for it partitions the possible delays into an ordered set of cohesive, contiguous regions. If the prior is, e.g., uniform; $\boldsymbol{g}^{-1} = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$, then, 'shortest delays' corresponds with an updated posterior distribution $\tilde{\boldsymbol{g}}^0 = (1/3, 1/3, 1/3, 0, 0, 0)$ and 'longest delays' corresponds with the updated distribution $\tilde{\boldsymbol{g}}^0 = (0, 0, 0, 1/3, 1/3, 1/3)$. With this system design, customers with service start times $\omega = 3$ will learn that their delay is one of the shortest and their utility be determined by the following sequence of updated distributions: $\boldsymbol{g}^{-1} = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$, $\boldsymbol{g}^0 = (1/3, 1/3, 1/3, 0, 0, 0)$, $\boldsymbol{g}^1 = (1/2, 1/2, 0, 0, 0)$, and $\boldsymbol{g}^2 = (1, 0, 0, 0)$ in periods $t = -1$, 0, 1 and 2, respectively, before starting service at $t = 3$.

Finally note that by the end of each period, when customer utility will be calculated, there is no information asymmetry between the service provider and the customer–both have complete information about the delay distribution, as in Ely et al. (2015). Therefore, there are a variety of alternative descriptions of the real-world dynamics that are consistent with the model. For example, we could assume that the customer observes outcome m directly and updates her own delay distribution. Or, we could describe m as both an outcome from the information system and a *message* sent from the service provider to the customer, and the Bayesian customer would then update the distribution, given the prior and received message. This latter framing of the problem would assume that the customer has complete knowledge of the provider's information system and message design and that the provider credibly commits to a messaging strategy.

3.2. Customer Utility

Utility will be calculated from the posterior means and standard deviations of the sequence of delay distributions. Specifically, for a customer still waiting at time t and given all distributions up to time t, the posterior expected start-time is:

$$v^t(\tilde{\boldsymbol{g}}^t) = \sum_{\tau=t+1}^{\bar{\omega}} \tau \tilde{g}^t_\tau. \tag{2}$$

A property of $v^t(\tilde{\boldsymbol{g}}^t)$ is that it is a martingale and therefore $\mathbb{E}_\Omega\left[v^t(\tilde{\boldsymbol{g}}^t)\right] = v^{-1}$ for $0 \le t \le \bar{\omega}$. That is, the average over all customers is always equal to the prior mean.

The standard deviation of the customer's posterior distribution at time t, $\tilde{\boldsymbol{g}}^t$, is:

$$\sigma^t(\tilde{\boldsymbol{g}}^t) = \sqrt{\sum_{i=t+1}^{\bar{\omega}} \tilde{g}^t_i (i - v^t(\tilde{\boldsymbol{g}}^t))^2}. \tag{3}$$

Now we describe how changes in delay distributions affect customers' utility in two ways: loss aversion and risk consciousness.

Loss aversion: We assume that customers have asymmetric preferences about changes in the *expected* service start-time. At each time $t \ge 0$, if the customer learns that her service will start later than she expects, she becomes anxious and her utility decreases, while if she learns that her service will start sooner than expected, she is pleased and her utility increases. Loss aversion implies that the utility penalty due to an increase in expected waiting-time is larger than the benefit from a same-size decrease in expected waiting time. This model is motivated by the initial concept of loss aversion as described by behavioral economists (Tversky and Kahneman 1991) and by more recent research showing how changes in beliefs can affect utility (Loewenstein and Molnar 2018), as well as empirical evidence for loss aversion among customers of ride-hailing systems and call centers (Yu et al. 2020, 2021).

Total utility is calculated by accumulating the instantaneous utilities that she experiences due to changes in beliefs during her wait (Kahneman et al. 1997). Thus, we assume that the impact of the changes in the expected service start-time on customer utility is additive over time, which is consistent with models in the behavioral operations and information design literature (Das Gupta et al. 2015, Ely et al. 2015).

Specifically, let $\delta = v^{t-1}(\tilde{\boldsymbol{g}}^{t-1}) - v^t(\tilde{\boldsymbol{g}}^t)$ be the decrease in expected service start-time between two adjacent time periods. We assume that the per-period utility is a function of δ: $u(\delta) = \delta$ for $\delta \geq 0$ and $u(\delta) = \lambda\delta$ for $\delta < 0$, where $\lambda > 1$ is the loss aversion penalty, similar to Gaur and Park (2007), Aflaki and Popescu (2013), and Long and Nasiry (2014). Therefore, the total expected utility for a loss-averse customer with service start-time ω and a given sequence of expected delays is:

$$U^L_\omega(\tilde{\boldsymbol{g}}_\omega) = \sum_{t=0}^{\omega-1} u(v^{t-1}(\tilde{\boldsymbol{g}}^{t-1}) - v^t(\tilde{\boldsymbol{g}}^t)) + u(v^{\omega-1}(\tilde{\boldsymbol{g}}^{\omega-1}) - \omega). \tag{4}$$

The last term captures the moment the service starts at $t = \omega$, and the customer learns the start time ω perfectly.

Risk consciousness: For some customers negative utility will be generated by uncertainty in the delay. Uncertainty may increase anxiety, affect a customer's mood, and increase perceived waiting time (Maister 1985, Larson 1988). Uncertainty may also have a concrete impact on customers, e.g., it may diminish the customer's ability to make reliable plans for when the service is over. We define the total expected utility from uncertainty for a customer with service start time ω and a given outcome to be the sum of the posterior standard deviations,

$$U^\sigma_\omega(\tilde{\boldsymbol{g}}_\omega) = -\sum_{t=0}^{\omega-1} \sigma(\tilde{\boldsymbol{g}}^t). \tag{5}$$

Following Anunrojwong et al. (2020), we call this aspect of utility *risk consciousness*. Similar models have been derived for risk-measurement in portfolio theory (Markowitz 1952, Kroll et al. 1984), where common measures of utility capture the variance (or standard deviation) of returns.

We define the total utility for a customer with service time ω and a given series of delay distributions as

$$U_\omega(\tilde{\boldsymbol{g}}_\omega) = U^L_\omega(\tilde{\boldsymbol{g}}_\omega) + \beta U^\sigma_\omega(\tilde{\boldsymbol{g}}_\omega), \tag{6}$$

where β is the weight given to risk consciousness. If $\beta = 0$ then the customer is purely loss averse. We will use the shorthand "$\beta = \infty$" for when the customer is not at all loss averse or when the customer's relative level of risk consciousness is so high so that the customer's loss aversion can be ignored. Finally, the service provider's objective is to maximize the expected utility, as in optimization problem (1).

Note that our objective, defined by (1) and (6), does not include a direct cost for waiting, e.g., the opportunity cost of delay. Also, the objective does not include a direct cost for expected delay, e.g., a cost representing the emotional toll of anticipating a long wait. These do not appear in the objective because both quantities are invariant to the posterior distributions. First, the updated distributions (customer beliefs) do not change the actual waiting-time. Second, the expected waiting-time, given any outcome, is a martingale (see above) and therefore its expected value across all customers is always equal to the prior v^{-1}. As long as the cost of anticipated waiting is linear in the expected duration of the wait, we can leave it out of the objective.

To reduce the notational burden, we occasionally drop the dependency on $\tilde{\boldsymbol{g}}_\omega$ from the arguments in $v^t(\tilde{\boldsymbol{g}}_\omega)$ and use $\tilde{v}^t$ instead. Similarly $\tilde{U}^L_\omega$ and $\tilde{U}^\sigma_\omega$ stand for $U^L_\omega(\tilde{\boldsymbol{g}}_\omega)$ and $U^\sigma_\omega(\tilde{\boldsymbol{g}}_\omega)$.

3.3. Numerical Example

To illustrate, we calculate the posterior means, standard deviations and utilities for customers exposed to a single announcement based on an information system with structure $AAABBB$. Assume that customers have a symmetric prior service start-time distribution $\boldsymbol{g}^{-1} = (1/12, 2/12, 3/12, 3/12, 2/12, 1/12)$, i.e., service starts at $\omega = 1$ with probability $1/12$, at $\omega = 2$ with probability $2/12$, etc. Given the structure $AAABBB$ we first calculate the posterior probabilities, expectations, and total loss aversion $\tilde{U}^L_\omega$ for each customer. Then we consider the evolution of standard deviation over time and calculate utility due to risk consciousness for each customer.

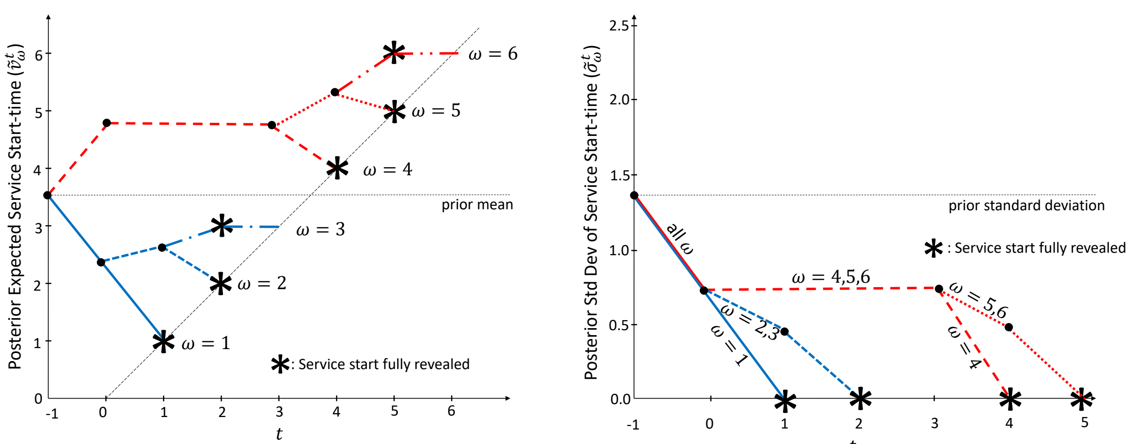


Figure 2 **(a) Expectation paths of $v^t(\tilde{\boldsymbol{g}}^t)$. Blue paths are for customers with outcome A and red paths are for customers with outcome B (left); (b) Paths of posterior standard deviation $\sigma^t(\tilde{\boldsymbol{g}}^t)$. Blue paths are for customers with outcome A and red paths are for customers with outcome B (right).**

Figure 2(a) shows how the customers' expected start-times evolve, given the $AAABBB$ structure. The horizontal axis shows the time t and the vertical axis represents the posterior expected service start-times, $\tilde{v}^t$. Each line in the plot represents *expectation paths* of posterior expected service start-times for each possible service start-time ω. Note that many expectation paths overlap and each path ends at the realized ω. To determine the path for a customer, start at the end of the path and trace backwards. Paths are colored blue and red, to indicate whether a customer had outcome A or B, respectively.

On the left-hand side of Figure 2(a), the initial expected service start-time when $t=-1$ is $v^{-1}=3.5$, the mean of the prior distribution. At time $t=0$ customers with outcome A learn that their start time is in the set $\{1,2,3\}$ and receive an updated forecast distribution according to equation (2). For example, given outcome A, $\tilde{g}_1^0=(1/12)/(1/12+2/12+3/12)=1/6$ and the posterior expected start-time for any customer with outcome A is $\tilde{v}^0=(1\times 1/12+2\times 2/12+3\times 3/12)/(1/12+2/12+3/12)=2.\overline{3}$. In Figure 2(a), this update is represented by the solid line that moves from the point $(-1,3.5)$ to the point $(0,2.\overline{3})$. Similarly, customers with outcome B have a new posterior expected value $\tilde{v}^0=4.\overline{6}$ that is calculated from the condition that they must have a start time in the set $\{3,4,5\}$.

Now consider how expected start times are updated when we move to $t=1$. With no additional announcement, the only new information is the passage of time. Customers with $\omega=1$ go into service and therefore their start time is fully revealed. This also reveals information to customers with $\omega=2$ or 3: they now know that they do not have $\omega=1$. They update their expected start time appropriately: given outcome A at $t=0$ and not going into service start time at $t=1$, $\tilde{v}^1=(2\times 2/12+3\times 3/12)/(2/12+3/12)=2.6$, as shown in Figure 2(a) by the dotted line that rises from $(0,2.\overline{3})$ to $(1,2.6)$. For customers with outcome B, however, moving from $t=0$ to $t=1$ does not convey any information, and therefore their expected start time does not change.

Now move from $t=1$ to $t=2$. Customers with $\omega=2$ go into service, fully revealing their start time. Because outcome A only included those customers in the set $\{1,2,3\}$, reaching $\omega=2$ also fully reveals the start time of those with $\omega=3$, because there is no other possible start times left in the set. Meanwhile, the posterior expected start time for those with outcome B again does not change. For time periods $t=4,5,6$ a similar sequence of updates and revelations occurs for those with outcome B.

Now we calculate the utility from loss aversion, $\tilde{U}^L_\omega$, for each customer. For numerical experiments throughout this paper we will assume that the loss aversion penalty is $\lambda=2$, a value suggested by estimates of analogous parameters (Tversky and Kahneman 1992). First consider the total utility for a customer with $\omega=1$. Following the expectation path for this customer in Figure 2(a), when moving from $t=-1$ to $t=0$, there is a drop in expected start time of service from 3.5 to $2.\overline{3}$. This implies a gain in utility of $v^{-1}-\tilde{v}^0=3.5-2.\overline{3}=1.1\overline{6}$. Similarly, from $t=0$ to $t=1$, there is another drop in expected delay and a gain of $2.\overline{3}-1=1.\overline{3}$. Therefore, the total utility for those with $\omega=1$ is $1.1\overline{6}+1.\overline{3}=2.5$, which is simply the difference between the prior mean 3.5 and the ultimate service time $\omega=1$.

Now consider customers with $\omega=2$. As for $\omega=1$, when moving from $t=-1$ to $t=0$, the customer picks up utility $3.5-2.\overline{3}=1.1\overline{6}$. From $t=0$ to $t=1$, however, the expected start time *increases*, so that there is a loss in utility, $\lambda(\tilde{v}^0-\tilde{v}^1)=2\times(2.\overline{3}-2.6)=-0.5\overline{3}$. Finally, when moving from $t=1$ to $t=2$ there is a utility gain of $2.6-2=0.6$, for a total utility of $1.1\overline{6}-0.5\overline{3}+0.6=1.2\overline{3}$. To evaluate total expected utility across all customers, we calculate the total utility along each expectation path for each value of ω as in equation (4) and then find the overall expected utility across all ω. For the ordinal single announcement with structure $AAABBB$, the expected utility due to loss aversion, $\mathbb{E}_\Omega\{\tilde{U}^L_\Omega\}=-1.02$.

Next, we calculate the posterior standard deviations and utility loss due to risk consciousness, $\tilde{U}^\sigma_\omega$. As shown in Figure 2(b), all customers begin with a prior standard deviation (SD) of 1.4. Given the outcomes at $t=0$, customers learn whether they will have a short wait $\{1,2,3\}$ or long wait $\{4,5,6\}$, which reduces their SDs to 0.75. This is the posterior SD at $t=0$ given outcomes A or B because both the information structure and prior start time distribution are symmetric. At $t=1$, customers with $\omega=1$ go into service, reducing their SD to 0, and the remaining customers with outcome A now know that they have either $\omega=2$ or 3, reducing their SD to 0.49. At $t=2$, both of these customers learn their true service start times. The evolution of SDs for $\omega=3,4,5$ are similar.

To calculate $\tilde{U}^\sigma_\omega$, we do not include the prior SD σ^{-1}, for this is independent of the information structure. Therefore, the overall utility due to risk consciousness for a customer with $\omega=1$ is -0.75 and for $\omega=2$ is $-(0.75+0.24)=-0.99$. Finally, the overall expected utility due to risk consciousness, $\mathbb{E}_\Omega\{\tilde{U}^\sigma_\Omega\}=-2.19$.

Now that we have demonstrated how to calculate the components of customer utility for the $AAABBB$ structure, we find the *optimal* structure by calculating utility for all $2^{\bar{\omega}-1}=32$ possibilities. We also calculate the MSE, $\mathbb{E}_\Omega\{\big(\Omega-v^0(\tilde{\boldsymbol{g}}^0)\big)^2\}$, a more traditional measure of forecast quality. Table 1 shows the optimal structure for each objective. The MSE-optimizing structure is $AAABBB$, but this structure does not maximize either component of customer utility. Loss aversion utility is maximized by $AABBAA$, and we call this an *onion* structure, for there are layers of distinct outcomes to the left and right of a central outcome, as in an onion.[3] Risk conscious

[3] Onion information structures might seem improbable, but they are seen in services. For example, upon arrival and before definitive diagnosis, an ED patient with chest pain may be having a heart attack (or suffer from angina) and a long ED stay is likely, or it may be Gastroesophageal reflux disease (heartburn) and therefore discharge will be quick. A building contractor may find that the cost and time to remove an internal wall to open up a kitchen might be very small or very large, depending on the location of internal support beams. Therefore, some customers may learn that they are in the tail of the cost distribution, without specifying which tail, while others learn that they are in the middle.

Objective	Optimal structure	MSE	$\mathbb{E}_\Omega\{\tilde{U}^L_\Omega\}$	$\mathbb{E}_\Omega\{\tilde{U}^\sigma_\Omega\}$
MSE, $\mathbb{E}_\Omega\{\left(\Omega - v^0(\tilde{\boldsymbol{g}}^0)\right)^2\}$	AAABBB	0.56	-1.02	-2.19
Loss Aversion, $\mathbb{E}_\Omega\{\tilde{U}^L_\Omega\}$	AABBAA	1.92	-0.72	-2.75
Risk Consciousness, $\mathbb{E}_\Omega\{\tilde{U}^\sigma_\Omega\}$	AAAABB	0.80	-0.98	-2.11

Table 1 Optimal structures for the numerical example

utility is maximized by $AAAABB$, with a partition for the longer delays (BB) that is smaller than the partition for the shorter delays ($AAAA$). While these are the optimal designs for a particular, symmetric prior distribution, we are interested in the optimal designs for other priors and other information cardinalities ($M > 2$). We will examine these questions in the analysis and numerical experiments in the following Sections.

4. Analysis for a Single Forecast Announcement

The general model with an arbitrary prior delay distribution and customers who are both loss averse and risk conscious is intractable for analysis (e.g., see the discussion in Section 4.2). Therefore, in the following subsections we examine two special cases. Section 4.1 analyzes the case when there are two information system outcomes, 'A' and 'B', and three possible delay realizations. Despite its simplicity, analysis of this model generates a number of insights about the optimal information structures, insights that hold when we examine numerical experiments for more general cases. To obtain further structural insights, in Section 4.2 we develop and analyze a continuous time model with two information system outcomes and loss averse customers.

4.1. Analysis with Three Periods ($\bar{\omega} = 3$)

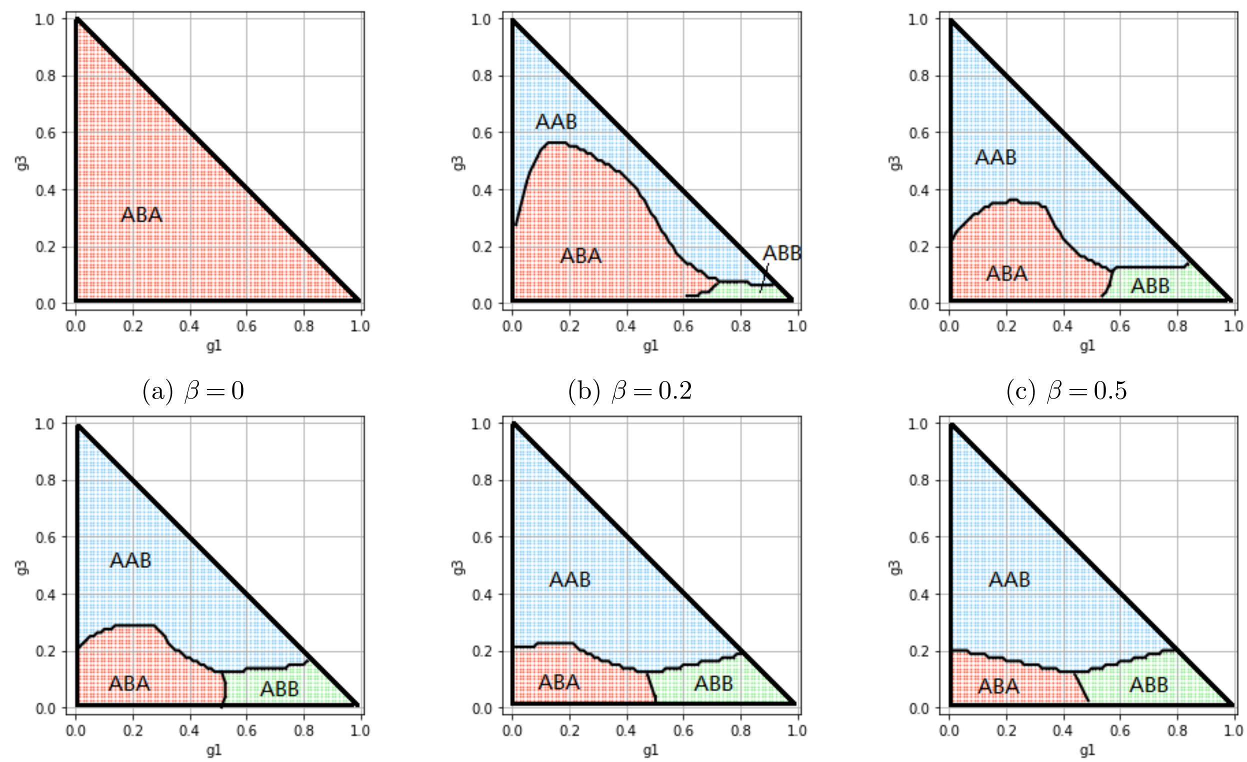
Here we will examine the optimal information structure, given that $\bar{\omega} = 3$. The three possible delays ω will be labelled $\{1, 2, 3\}$, with prior probabilities $(g_1^{-1}, g_2^{-1}, g_3^{-1})$ respectively, and these may be interpreted as short, medium and long delays. There are three possible information structures at $t = 0$: AAB, ABB, and ABA.[4] If customers are purely loss averse, we show that an ABA structure is always optimal.

PROPOSITION 1. *Assume that there is a single, initial announcement at $t = 0$. When $\bar{\omega} = 3$, $\beta = 0$ (customers are loss averse), and the number of outcomes $M = 2$, the information structure ABA is optimal for any prior distribution and any $\lambda > 1$.*

Proposition 1 is remarkable in the sense that for loss-averse customers with three periods and two information outcomes, an ordinal structure is never optimal. The ABA onion information structure distinguishes customers with very short or very long delays from those with moderate delays; The onion structure is optimal because it flattens the expectation paths of the customers.

[4] Or, equivalently, structures BBA, BAA, and BAB.

That is, the true service time is revealed for customers with $\omega = 2$, while for customers with $\omega = 1$ and $\omega = 3$ the posterior expectation remains close to the prior mean until the actual service times are revealed at $t = 1$. This minimizes the degree of unwarranted hope, followed by disappointment, experienced by these loss averse customers.



(a) $\beta = 0$ (b) $\beta = 0.2$ (c) $\beta = 0.5$

(d) $\beta = 0.8$ (e) $\beta = 2$ (f) $\beta = \infty$

Figure 3 **Optimal information structures for a single announcement with $\bar{\omega} = 3$, as β varies**

If customers are purely risk conscious, $\beta = \infty$, we find that any of the three possible structures may be optimal and that there are linear boundaries between the regions of optimality.

Proposition 2. *Assume that there is a single, initial announcement at $t = 0$. When $\bar{\omega} = 3$, $\beta = \infty$ (customers are risk conscious), and number of outcomes $M = 2$, with prior $\boldsymbol{g}^{-1} = (g_1^{-1}, 1 - g_1^{-1} - g_3^{-1}, g_3^{-1})$. The optimal information structure is:*

- *AAB when $g_1^{-1} \leq 4g_3^{-1}$ and $g_1^{-1} + 5g_3^{-1} \geq 1$*
- *ABA when $g_1^{-1} + 5g_3^{-1} \leq 1$ and $2g_1^{-1} + g_3^{-1} \leq 1$*
- *ABB when $g_1^{-1} \geq 4g_3^{-1}$ and $2g_1^{-1} + g_3^{-1} \geq 1$*

According to Proposition 2, ordinal strategies may be optimal for risk conscious customers, but not always.

Plot (f) in Figure 3 illustrates this result. In this plot the x-axis is g_1^{-1}, the y-axis is g_3^{-1}, and each plot shows all values for the prior such that $g_1^{-1} + g_3^{-1} \leq 1$ (we always set $g_2^{-1} = 1 - g_1^{-1} - g_3^{-1}$). In the lower left-hand corner of Figure 3(f), when service is unlikely to start very early or very late, i.e. g_2^{-1} is large, the onion structure is again optimal. This is because the ABA structure reveals the service time for customers with $\omega = 2$, eliminating all uncertainty for those with the most likely delay. Likewise, in the lower right when g_1^{-1} is large, the optimal structure is ABB so that those with $\omega = 1$ are immediately aware that their delay will be short. Overall, however, the AAB structure has the largest optimality region; this information structure immediately reveals the status of those with the longest delay, $\omega = 3$. This is consistent with the example in Table 1 and the numerical experiments of Section 5; we will see that when customers are risk conscious it is often optimal to provide the most precise information to customers with the longest wait times.

Subplots (b) - (e) of Figure 3, generated numerically, show how the optimal information structure changes as β increases. The ABA structure that is always optimal when customers are purely loss averse gives way to ordinal structures for risk conscious customers. Section 4 in the Appendix presents the optimal information structures for $\bar{\omega} = 4$. There we see similar patterns, e.g., the dominance of onion structures for low values of β and ordinal structures for high β.

4.2. Analysis in Continuous Time for Loss Aversion

To obtain additional structural insights, we develop and analyze a continuous version of the discrete problem (1) with $\beta = 0$. We show that for two information outcomes and for a given set of prior distributions, the optimal information structure is simple: either AB (ordinal) or ABA (onion). We show that for this set of prior distributions, more complex information structures, such as $ABAB$ or $ABABA$, are not optimal. We also show that if the optimal structure is ABA, then the posterior mean of the updated forecast is equal to the prior mean for each and every customer. The analysis of the continuous model also highlights how the optimal structure balances customer utility when the forecast is first sent, as time passes, and when service begins.

Consider a system with loss averse customers, with a maximum delay $\bar{\omega}$, and a single announcement when customers arrive. There are no additional announcements after $t = 0$, and two information outcomes are possible, $m \in \{A, B\}$. We first rewrite the discrete-time objective function of problem (1) to show that customer utility is separable into time-based components, which we label the beginning, middle and end effects. We then formulate this objective in continuous time and show that it is a constrained maximization problem over a *convex* function. This formulation allows us to show that for a particular class of prior distributions, the optimal solution is at a corner point.

Recall that $\tilde{v}^t$ is the ex ante expected start time of service provided that the customer did not start service yet in period t. $\tilde{v}^t$ depends on the outcome m, via the information structure. Let

p_m be the probability of outcome m for any given customer and v_m^t be the posterior *conditional* expected value in time period t after generating outcome m. For $t \geq 1$ and any outcome m, the expected conditional start time of service only rises with the passage of time. That is, $v_m^t - v_m^{t-1} \geq 0$ for $1 \leq t \leq \omega - 1$. In addition, for a customer who goes into service at $t = \omega$, $v_m^{\omega-1} - \omega \geq 0$ because when the customer reaches $\omega - 1$ without going into service, the expected remaining delay is at least one period in the future. Therefore, we can rewrite the expected utility due to loss aversion, the expected value of expression (4) as:

$$\sum_{\omega=1}^{\overline{\omega}} \sum_{m\in\{A,B\}} f_\omega(m)\Big(\underbrace{\{v^{-1} - v_m^0\}^+ - \lambda\{v_m^0 - v^{-1}\}^+}_{\text{beginning effect, } v^{-1}-v_m^0-(\lambda-1)\{v_m^0-v^{-1}\}^+} - \underbrace{\lambda\sum_{\tau=1}^{\omega-1}(v_m^\tau - v_m^{\tau-1})}_{\text{middle effect, } v_m^\tau > v_m^{\tau-1}} + \underbrace{(v_m^{\omega-1} - \omega)}_{\text{end effect, } v_m^{\omega-1} > \omega} \Big) g_\omega^{-1} \quad (7)$$

The three terms on the right-hand-side in (7) correspond to what we call the *beginning*, *middle*, and *end* effects. The beginning effect in (7) is due to the change in the prior after the outcome is observed at $t = 0$ via Bayes' rule. After this initial update, the prior is revised upward as time passes, until it reaches $\tilde{v}^{\omega-1}$, representing a penalty for these loss-averse customers; that is the middle effect, the second term in (7). The end effect, the third term in (7), always contributes positive utility as the customer receives the good news that she is going into service.

We will rewrite equation (7) to show that the utility can be expressed as an even simpler trade-off between the loss of utility at $t = 0$ due to the initial announcement vs. the duration of expected delay just before going into service, which determines the intensity of the positive surprise at the end of the delay. The overall customer experience depends on how the initial forecast balances these two quantities. Notice that as time progresses and the posterior expected value always increases over time, the cumulative effect, which is the middle effect, depends only on the difference between the posterior at time $t = 0$ and time $t = \omega - 1$, i.e., right before entering service; $\lambda(v_m^0 - v_m^{\omega-1})$. Using the martingale property of Bayesian posteriors, $\sum_{\omega=1}^{\overline{\omega}} \sum_{m\in\{A,B\}} f_\omega(m) v_m^0 g_\omega^{-1} = v^{-1}$, we can simplify equation (7) further:

$$\begin{aligned}
&\sum_{\omega=1}^{\overline{\omega}} \sum_{m\in\{A,B\}} f_\omega(m)\left[v^{-1} - v_m^0 - (\lambda-1)\{v_m^0 - v^{-1}\}^+ + \lambda(v_m^0 - v_m^{\omega-1}) + (v_m^{\omega-1} - \omega)\right] g_\omega^{-1} \\
=&\underbrace{\sum_{\omega=1}^{\overline{\omega}} \sum_{m\in\{A,B\}} f_\omega(m)[v^{-1} - v_m^0] g_\omega^{-1}}_{=0 \text{ (martingale property)}} - (\lambda-1)\sum_{\omega=1}^{\overline{\omega}} \sum_{m\in\{A,B\}} f_\omega(m)[\{v_m^0 - v^{-1}\}^+ + v_m^{\omega-1}] g_\omega^{-1} \\
&+\underbrace{\sum_{\omega=1}^{\overline{\omega}} \sum_{m\in\{A,B\}} f_\omega(m)[\lambda v_m^0 - \omega] g_\omega^{-1}}_{=(\lambda-1)v^{-1} \text{ (martingale property)}}
\end{aligned}$$

$$= -(\lambda-1)(\sum_{\omega=1}^{\bar{\omega}} \sum_{m\in\{A,B\}} f_\omega(m)[\{v_m^0 - v^{-1}\}^+ + v_m^{\omega-1}]g_\omega^{-1} - v^{-1}) \tag{8}$$
$$= -(\lambda-1)(\underbrace{\mathbb{E}_{\tilde{v}^0}\{\{\tilde{v}^0 - v^{-1}\}^+\}}_{\text{expected beginning effect}} + \underbrace{\mathbb{E}_{\tilde{v}^{\Omega-1}}\{\tilde{v}^{\Omega-1} - v^{-1}\}}_{\text{expected end surprise}})$$

With this formulation, notice that when there is no loss aversion, i.e. when $\lambda = 1$, the expected utility under *any* information structure is equal to zero. This is intuitive as due to the martingale property of posterior beliefs, in expectation, the posterior increases as much as it decreases. If equal weight is associated with increases and decreases, in expectation, these cancel out. Note also that because $-(\lambda-1)$ can be factored out of the objective, as long as $\lambda > 1$ the optimal solution to the maximization problem does not depend on the specific value of λ.

From this formulation we see that the objective is to minimize the sum of the beginning effect, the expected value of $\{\tilde{v}^0 - v^{-1}\}^+$, and the expected value of the end surprise, *when entering service*, $\tilde{v}^{\Omega-1} - v^{-1}$. While this formulation is simple to interpret, it is still challenging to analyze and develop insights into the problem. Even with a single announcement and just two outcomes, there is an exponential number of strategies to consider ($2^{\bar{\omega}-1}$). Therefore, for analytic tractability we replace the prior discrete distribution over Ω with a continuous density over $[1,\bar{\omega}]$ with density $g(\omega) > 0$ and cumulative distribution function $G(\omega)$. The continuous formulation allows us to derive structural properties of the problem. Now, $v^{-1} = \int_1^{\bar{\omega}} \omega g(\omega)d\omega$ is the prior expected start-time. Then, instead of decision variables $\boldsymbol{f}$, the decision variable is a function $f_m(\omega) : [1,\bar{\omega}] \longmapsto [0,1]$ for $m \in \{A,B\}$ with $f_A(\omega) + f_B(\omega) = 1$ for all $\omega \in [1,\bar{\omega}]$. The analogue of equation (8), formulated in continuous time, is,

$$-(\lambda-1)\left(\int_1^{\bar{\omega}} \sum_{m\in\{A,B\}} [\{v_m^0 - v^{-1}\}^+ + (v_m(\omega) - v^{-1})]f_m(\omega)g^{-1}(\omega)d\omega\right) \tag{9}$$
$$\text{with } v_m(\omega) = \frac{\int_\omega^{\bar{\omega}} \tilde{\omega} f_m(\tilde{\omega})g^{-1}(\tilde{\omega})d\tilde{\omega}}{\int_\omega^{\bar{\omega}} f_m(\tilde{\omega})g^{-1}(\tilde{\omega})d\tilde{\omega}} \text{ and } v_m^0 = v_m(1)$$

This infinite-dimensional optimization problem, with maximization over the function $f_m(\omega)$, is also challenging. We follow the the approach of Kamenica and Gentzkow (2011) and re-formulate the problem in terms of the *posterior* (tail) distribution of the start time of service, after generating an outcome. Let $p_m = \int_1^{\bar{\omega}} f_m(\tilde{\omega})g^{-1}(\tilde{\omega})d\tilde{\omega}$ be the ex ante probability that the customer generates outcome m and $\boldsymbol{p} = (p_A, p_B)$, $p_A + p_B = 1$. The prior expected value can be expressed as a function of the prior tail distribution, $\bar{G}(\omega) \triangleq 1 - G(\omega)$ and $v^{-1} = 1 + \int_1^{\bar{\omega}} \bar{G}(\omega)d\omega$. With two outcomes $m \in \{A,B\}$, we denote with a slight abuse of notation, $\bar{G}_m(\omega)$ as the posterior tail distribution and $v_m^0 \triangleq 1 + \int_1^{\bar{\omega}} \bar{G}_m(\omega)d\omega$ as the posterior expected value, right after outcome $m \in \{A,B\}$ at time

$t=0$. For any given $\boldsymbol{p}$ and $(\bar{G}_A(\omega), \bar{G}_B(\omega))$, the information structure $f(\omega)$ is uniquely determined and vice versa.

In equation (9), the posterior immediately after generating outcome m at $t=0$, v_m^0, depends in a straightforward way on $\bar{G}_m(\omega)$. The posterior at any time $t=\omega$, which we denoted as $v_m(\omega)$, however, is non-trivial. Proposition 3 provides an explicit formulation that will allow us to obtain insights as to how the information structure affects customer utility.

PROPOSITION 3. *The continuous optimization problem (9) for loss averse customers can be written as:*

$$\max_{\boldsymbol{p},\boldsymbol{G}} \; -(\lambda-1)\times\left(\sum_{m\in\{A,B\}} p_m\left[\underbrace{\{1+\int_1^{\bar{\omega}}\bar{G}_m(\omega)d\omega}_{=v_m^0}-v^{-1}\}^+ +\int_1^{\bar{\omega}} T(\bar{G}_m(\omega))d\omega\right]-v^{-1}\right) \; s.t. \; (\boldsymbol{p},\boldsymbol{G})\in\mathcal{G}.$$

where $T(G) = -G\ln(G)$ *and* $\mathcal{G} \triangleq \{(\boldsymbol{p},\boldsymbol{G})$ *such that* $\bar{G}_m(\omega)$ *is continuous non-increasing with* $\bar{G}_m(1) = 1$ *and* $\bar{G}_m(\bar{\omega}) = 0$ *and for which* $\sum_{m=1}^{M} p_m(\bar{G}_m(\omega) - \bar{G}(\omega)) = 0$ *for all* $\omega \in [1,\bar{\omega}]$ *and* $\sum_{m\in\{A,B\}} p_m = 1$.

Here, $\mathcal{G}$ is the set of all posterior tail distributions of the start time that can be generated via a system with two outcomes $\{A,B\}$ and for which the probability of these outcomes is $\boldsymbol{p}$.

For a given $\boldsymbol{p}$, Proposition 3 highlights the structure of the (continuous) maximization problem: The constraints of $\mathcal{G}$ are linear in the posterior tail distribution $\bar{G}_m(\omega)$. As $-T(G)$ is convex in G in the second term of the objective that is to be maximized, the standard solution approach using Pontryagin's maximum principle cannot be applied.

To obtain insights into the structure of the optimal solution, consider a partitioning of $[1,\bar{\omega}]$ into K segments via thresholds $\boldsymbol{\omega}=(\omega_1,\omega_2,...,\omega_{K-1})$ with $1=\omega_0<\omega_1<...<\omega_{K-2}<\omega_{K-1}<\omega_K=\bar{\omega}$. Without loss of generality, let outcome A be given in segment $[\omega_0,\omega_1)$, outcome B in $[\omega_1,\omega_2)$, A in $[\omega_2,\omega_3)$ and so on. For a given $\boldsymbol{p}$, *any pair* of continuous posterior tail distributions $\bar{G}_m(\omega)$ that satisfies $(\boldsymbol{p},\boldsymbol{G})\in\mathcal{G}$ can be approximated by

$$\bar{G}_m(\omega;\boldsymbol{\omega},\boldsymbol{p}) = \frac{\sum_{k:\text{outcome } m \text{ in}[\omega_{k-1},\omega_k)} \int_{\max(\omega,\omega_{k-1})}^{\max(\omega,\omega_k)} g(\omega)d\omega}{p_m},$$

for a large enough K and appropriate $\boldsymbol{\omega}$. Intuitively, we introduce an information structure $ABABABA$... over K intervals for a given $\boldsymbol{p}$, and we will first optimize over the thresholds $\boldsymbol{\omega}$ (and then over $\boldsymbol{p}$). While, in principle, we might need an infinite number of intervals ($K=+\infty$), the formulation of Proposition 3 below allows us to show that an information structure with a finite number of intervals is sufficient. It turns out that under certain conditions of the prior distribution, we need not consider more than three intervals ($K=3$).

Figure 4 illustrates the proof idea for $K = 6$. Assume that all intervals are non-empty; $\omega_{k-1} < \omega_k$ for $k = 1, ..., 6$. Next, we take two consecutive segments with outcome B—the shaded segments in Figure 4, $[\omega_1, \omega_2) \cup [\omega_3, \omega_4)$—and fix the probability mass in each of these segments, as well as the posterior mean, v_B^0 of the structure. Thus, we impose two conditions on the mass and one condition on the posterior expected value[5]. Therefore, the problem has three constraints. Out of four thresholds $(\omega_1, \omega_2, \omega_3, \omega_4)$, only one threshold, say ω_1, can be freely changed under these constraints. As the posterior mean and the probabilities in each segment are fixed, *increasing* ω_1 shifts *up* the lower segment and *down* the upper segment, or, vice versa, as indicated by the arrows in Figure 4. We show that, under some conditions of the prior and the aforementioned constraints, the objective (to be maximized) is convex in ω_1. This implies that to maximize the customer utility, either the middle segment A must be eliminated (the arrows that point to each other), or, one of the two adjacent segments A must be eliminated (arrows that point outward). Thus, it suffices to consider $\boldsymbol{\omega}$ with strictly fewer segments, as long as there are two B segments that can freely be adjusted under the three constraints, i.e. $K \geq 5$. Furthermore, we show that under one additional condition on the prior, the objective for $K = 4$ can further be improved by eliminating one interval. Therefore, under these conditions, for any given posterior expected value, the optimal solution must have three segments or less, i.e., $K \leq 3$. Finally, for $K = 3$, keeping only $\boldsymbol{p}$ fixed and also optimizing over the posterior expected value, i.e. the first term $\{\tilde{v}_m^0 - v^{-1}\}^+$, we show that either the optimal structure is onion, $K = 3$, with the posterior mean equal to the prior mean, or the optimal structure is ordinal, $K = 2$:

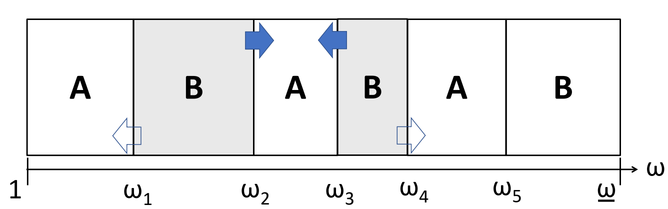


Figure 4 **Proof idea for $K = 6$. When changing ω_1, keeping the probabilities in the two shaded areas in which outcome B is given and keeping the posterior expected value, v_B^0 constant, the customer utility increases when either the A region sandwiched in between the two shaded areas is eliminated, or, one of the adjacent A regions is eliminated. Starting from any number of segments, K, the customer utility increases when eliminating one segment. Under the conditions on the prior in Proposition 4, this can be repeated until $K = 3$.**

[5] $\bar{G}(\omega_1) - \bar{G}(\omega_2) = \text{constant}_1$, $\bar{G}(\omega_3) - \bar{G}(\omega_4) = \text{constant}_2$ and $1 + \int_1^{\bar{\omega}} \bar{G}_m(\omega) d\omega = \text{constant}_3$ are fixed at some constants.

Proposition 4. *When $g(\omega)$ is concave and non-increasing, for loss averse customers, either an ordinal structure AB is optimal (i.e. $1 < w_1^* < \underline{\omega}$) with a beginning effect, $v_B^0 > v^{-1} > v_A^0$, or an onion structure ABA is optimal (i.e. $1 < w_1^* < \omega_2^* < \underline{\omega}$), for which the beginning effect is eliminated $v_A^0 = v_B^0 = v^{-1}$.*

The proposition highlights that the optimal structure is a trade-off between the beginning effect on the one hand and the middle and end effect on the other hand. The combined middle and end effect is represented by the term $v_m(\omega) - v^{-1}$ in equation (9). This corresponds to the expected surprise immediately before a customer goes into service. To maximize the overall objective, it can be optimal to make this surprise as small as possible, i.e. the forecast must be as precise as possible *right before the customer enters service.* Therefore structures with multiple distinct, separated regions across the delay space would be non-optimal. For example, an $ABABA$ structure spreads the information for customers who generate outcome B across two separate regions, so that the expectations of customers in the lower B region will 'overshoot' their actual delays. Therefore, it is optimal to combine those regions into an ABA structure. This is interesting, for it advises against the strategy of creating surprises by calling in customers earlier than they expect (e.g. when "under-promising and *over*delivering").

Given that it is optimal to combine regions, why does ABA (onion) sometimes dominate AB (ordinal)? Why not combine the two A regions into one? Here, the term $\{v_m^0 - v^{-1}\}^+$ in equation (9) comes into play. This is the beginning effect, and for some priors it is optimal to use an ABA structure to eliminate the beginning effect, as averaging the two A regions allow the posterior expectation to equal the prior for those customers who generate the A outcome. Eliminating the beginning effect is impossible with an ordinal structure for which one necessarily must have $v_B^0 > v^{-1} > v_A^0$, that is, at least some customer will be disappointed in the beginning. This intuition helps to explain the optimality of the onion structure in Proposition 1 with three possible delays ($\bar{\omega} = 3$) and the numerical experiments we discuss in the following section.

Corollary 1. *For the uniform distribution and for loss averse customers, outcomes A and B are given with equal probability: outcome A for the 25% percentile shortest and the 25% percentile longest wait times and outcome B for the 50% of delays in the middle. That is, $\omega_1^* = 1 + \frac{\bar{\omega}-1}{4}$ and $\omega_2^* = \bar{\omega} - \frac{\bar{\omega}-1}{4}$.*

The Corollary illustrates how the onion structure eliminates the beginning effect completely when the wait time density is uniform. In other words, the outcome is informative as it indicates whether the wait time is either 'extreme' or not, but does not change the prior expectation of wait time for customers who generate either outcome.

The conditions in Proposition 4 on the prior distribution, that $g(\omega)$ is concave and non-increasing, are satisfied by densities of the form $(k_0 - \int_1^{\omega} k(t)dt)$, with k_0 the normalization constant and $k(\omega)$ any non-negative, non-decreasing function over $[1, \bar{\omega}]$. A special case is when $k(\omega) = 0$, then $g(\omega)$ is the uniform distribution. We will see in the next Section that the insights derived from Proposition 4 apply to a wide variety of priors in addition to those that are concave and non-increasing. In addition, Proposition 4 assumes that there are two types of outcomes. By continuity, we expect that for more than two outcomes ($M \geq 2$), it is still optimal to eliminate the beginning effect with an onion information structure, and this is confirmed in the following numerical experiments. However, when many outcomes are possible (e.g. $M \geq K/2$) the onion structure is not feasible. Next we explore the optimal structure numerically for any number of outcomes, M, and prior distribution $g^{-1}(\omega)$.

5. Numerical Experiments with a Single Announcement

Motivated by the analysis in the previous section, the following numerical experiments focus on four questions: (i) what are the optimal information structures for loss averse and risk conscious customers? In particular, under what conditions are onion and ordinal structures optimal? (ii) How is the absolute level of customer utility affected by information structure choice, and what is the impact on utility when structures are restricted to be ordinal? (iii) Given an ordinal structure and a limited number of outcomes, some outcomes may be allocated to a smaller range of delay realizations; those outcomes will be more *precise*. In the optimal ordinal structure, which outcomes are most precise? (iv) How does the information structure affect the timing of changes in customer utility, e.g., what is the size of the beginning, middle and end effects under optimal strategies for loss averse and risk conscious customers?

Overall, our numerical study confirms the insights generated by the previous analysis and shows that these insights are robust with respect to the prior distribution and the number of outcomes. Throughout these experiments we assume that $\bar{\omega} = 7$. This is an $\bar{\omega}$ large enough to provide interesting results but small enough so that it is feasible to find optimal partitions for large numbers of examples.

5.1. Optimal Information Structures: Onion and Ordinal

First we examine how the optimal information structures change as the degree of customer loss aversion vs. risk consciousness varies and as the number of possible outcomes increases. Table 2 shows how the optimal information structure varies for a uniform prior as β and M vary (recall that β is the weight given to risk consciousness). The columns labelled "Structure" show the optimal information structures, with each distinct outcome represented by a different letter. For example, when $M = 3$ and $\beta = 0$, the optimal information structure is $ABCCCBA$, an onion structure with

	$\beta=0$				$\beta=0.3$				$\beta=2$			
M	Structure	$-U^L$	$-U^\sigma$	-Tot	Structure	$-U^L$	$-U^\sigma$	-Tot	Structure	$-U^L$	$-U^\sigma$	-Tot
1	AAAAAAA	1.50	5.61	1.50	AAAAAAA	1.50	5.61	3.18	AAAAAAA	1.50	5.61	12.72
2	ABBBBBA	1.14	4.01	1.14	ABBBBBA	1.14	4.01	2.35	AAAAABB	1.50	3.00	7.50
3	ABCCCBA	0.93	3.19	0.93	AABBBCC	1.07	2.19	1.73	AAAABBC	1.36	1.85	5.05
4	ABCDCBA	0.86	2.86	0.86	AABBBCD	1.00	1.34	1.40	AAABBCD	1.14	1.06	3.27
5	ABCDEBA	0.86	2.00	0.86	AABBCDE	1.00	0.57	1.17	AAABCDE	1.07	0.49	2.06
6	ABCDEFA	0.86	0.86	0.86	AABCDEF	0.93	0.14	0.97	AABCDEF	0.93	0.14	1.21
7	ABCDEFG	0.86	0.00	0.86	ABCDEFG	0.86	0.00	0.86	ABCDEFG	0.86	0.00	0.86

Table 2 Optimal Information Structures for $\beta=[0,0.3,2]$, Uniform Prior and $\bar{\omega}=7$.

outcome A generated by customers with start time $\omega\in\{1,7\}$, outcome B generated by $\omega\in\{2,6\}$, and outcome C generated by $\omega\in\{3,4,5\}$. The columns labelled $-U^L$ and $-U^\sigma$ show the total expected disutility due to loss aversion ($-\mathbb{E}_\Omega\{\tilde{U}^L_\Omega\}$) and risk consciousness ($-\mathbb{E}_\Omega\{\tilde{U}^\sigma_\Omega\}$) respectively. The columns labelled '-Tot' show the total disutility, $-\mathbb{E}_\Omega\{\tilde{U}^L_\Omega+\beta\tilde{U}^\sigma_\Omega\}$.

Consider the optimal information structures in the second column. In the previous analysis we saw that the onion structure is optimal when $M=2$, $\beta=0$ and $\bar{\omega}=3$ (Section 4.1) as well as for a particular class of continuous priors when $M=2$ (Section 4.2). Table 2 shows that the onion structure is also optimal when $\beta=0$ for $M=3$ and $M=4$, and the onion structure with $M=4$ achieves the maximum possible loss-averse utility. Note that the onion structure is only possible when $M\le\bar{\omega}/2+1$. Looking at the other side of the Table, when risk consciousness is dominant (with $\beta=2$), the ordinal structure is optimal. Finally, for the intermediate value $\beta=0.3$, the onion structure is optimal when $M=2$ but flips to an ordinal structure for larger values of M.

We now examine the optimal structures for other prior distributions. We generate 1,000 prior distributions by drawing values of $g^{-1}_\omega, \omega\in\{1,2,\ldots,7\}$ from a uniform distribution ($U[0.001,1]$) and then normalize the resulting probability mass function. Each of the resulting distributions has an arbitrary shape, but among the 1,000 are asymmetric distributions, as well as roughly symmetric, increasing, and decreasing distributions. For each randomly-generated distribution we find the optimal information structure. Tables 3 and 4 summarize the results. We see from Table 3 that when $\beta=0$ onion structures tend to dominate, but are not always optimal, when M is small. For larger M, other structures are optimal. From Table 4 we see that when customers are risk-conscious (e.g. large β), ordinal structures are almost always optimal.

While the prior distributions for these experiments can have any shape, we performed three sets of additional experiments with large numbers of priors that are explicitly constrained to be unimodel, increasing, and decreasing. For each of these sets of additional experiments the results are similar to those shown in Tables 3 and 4. To summarize the results of all of these experiments: onion structures are often optimal for loss averse customers when there are few information outcomes, and ordinal structures are virtually always optimal for risk conscious customers, no matter how many outcomes.

Number of outcomes (M)	2	3	4	5	6
Ordinal structure	0.02%	6%	10%	12%	22%
Onion structure	76.88%	50%	8%	0%	0%
Other structures	23.10%	44%	82%	88%	78%

Table 3 **Optimal structures for loss-averse customer ($\beta = 0$) and $\bar{\omega} = 7$.**

Number of outcomes (M)	2	3	4	5	6
Ordinal structure	99.68%	98%	96%	92%	88%
Onion structure	0.32%	0%	0%	0%	0%
Other structures	0%	2%	4%	8%	12%

Table 4 **Optimal structures for risk-conscious customers ($\beta = \infty$) and $\bar{\omega} = 7$.**

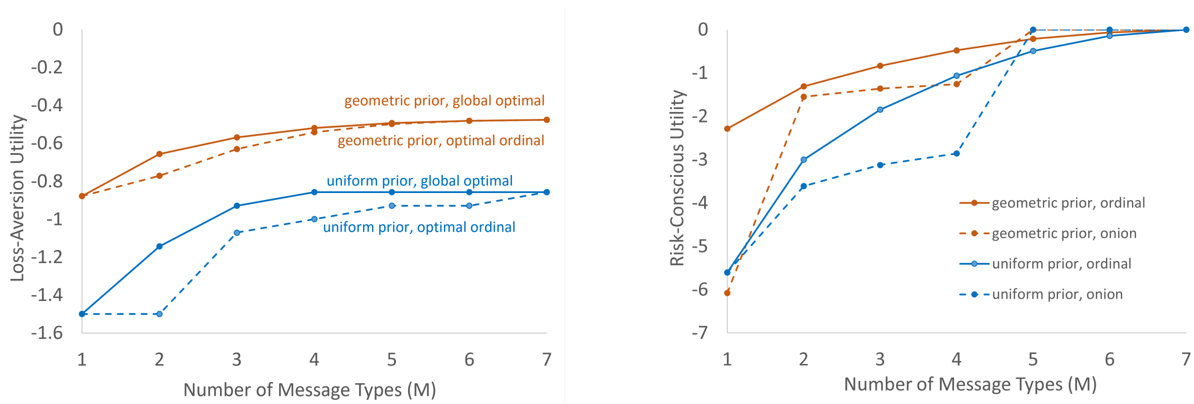


Figure 5 **(a) Optimal utility of loss-averse customers ($\mathbb{E}_\Omega\{\tilde{U}^L_\Omega\}$) as the number of possible outcomes (M) increases and $\bar{\omega} = 7$ (left); (b) Optimal utility of risk-conscious customers ($\mathbb{E}_\Omega\{\tilde{U}^\sigma_\Omega\}$) as M increases and $\bar{\omega} = 7$ (right).**

5.2. Impact of Information Structure on Utility

We now consider how the optimal information structure affects customer utility. Returning to Table 2, we see from the columns labeled $-U^L$ and $-U^\sigma$ that when $\beta = 0$ the onion structure reduces loss aversion but has relatively high uncertainty, as the posterior distribution can have a high variance. When β is large, the ordinal structure reduces disutility due to uncertainty, at the cost of increasing loss aversion. The columns labelled "-Tot" indicate that utility increases as the number of outcomes increases and that the increase can be roughly concave. To see this graphically, we plot the total utility for the optimal structures with $\beta = 0$ as the solid blue line in Figure 5(a). While Table 2 was generated using a uniform prior, Figure 5(a) shows additional examples of the impact of the number of outcomes M on utility, given a truncated geometric prior (a prior with diminishing probabilities for longer delays). We see similar, roughly concave increasing utility in M.

In addition to showing the utility generated under the optimal information structures, Figure 5(a) also shows utility for the optimal *ordinal* structures, for ordinal structures are frequently used in practice. In this Figure, $\beta = 0$, so that requiring ordinal structures result in some loss of utility, and that loss can be significant when the information system is coarse rather than granular (small M). A similar pattern can be seen when risk consciousness is dominant. Figure 5(b) shows the utility of strictly risk-conscious customers. Here again, utility increases with M. Again, when M is small, there can be a significant loss in utility when an optimal onion structure is used rather than the globally optimal ordinal structure. To summarize the results of these experiments: increasing the granularity of the information system provides decreasing returns. When the number of outcomes is small it is important to use the optimal information structure (onion or ordinal), but the loss in utility due to using the 'wrong' structure diminishes as the number of possible outcomes rises.

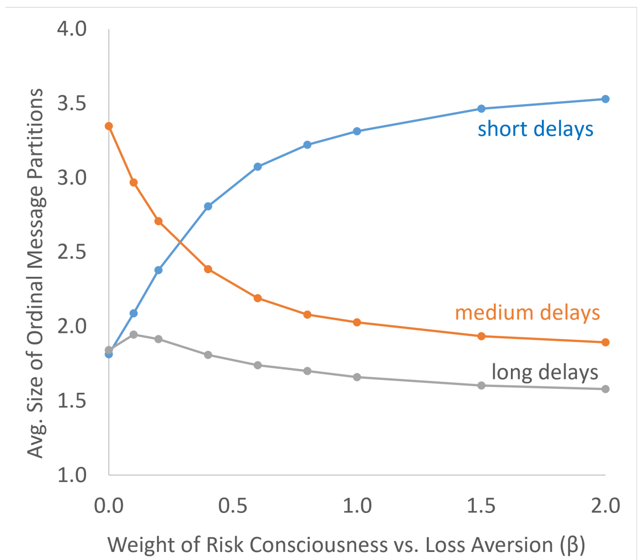


Figure 6 **Average size of optimal ordinal partitions with three outcomes ($M = 3$) for randomly-generated priors with $\bar{\omega} = 7$. The three partitions indicate short, medium and long delays; small partition size indicates narrow ranges and high precision while large partition size indicates wider ranges and low precision.**

5.3. Information System Precision

Given a limited number of outcomes and a larger number of delay realizations, an information structure may allocate a small number of realizations to some outcomes and a large number to others. When the information structure is ordinal, this implies that some partitions for particular outcomes will have a narrow range of delays. We say that an outcome is more *precise* if the size of the partition is smaller. We will begin examining precision for arbitrary optimal structures but then again focus on ordinal structures.

First, return to Table 2. For the onion structures in the second column, the outcomes on the tails are more precise, e.g., for $M = 2$ the optimal structure is $ABBBBBA$, so that the outcome indicating tail delays has a partition of size two and the middle has size five. On the right-hand-side of the table, for the ordinal structures designed for large β, the longer delays are placed into smaller partitions. That is, the optimal outcomes provide more precision for longer delays than for shorter delays. We saw a similar effect in §4.1 and Figure 3(f), where information structure AAB dominated when $\beta = \infty$: the outcome for the longest delay is more precise than the outcome for shorter delays.

Now we examine whether these patterns hold for any prior. To generate Figure 6 we hold $M = 3$, and for each of 1,000 randomly-generated priors we find the optimal *ordinal* structure. Then, we average the sizes of the partitions, A ("short delays"), B ("medium delays") and C ("long delays"). We see from the figure that when risk consciousness is dominant (β is high), long delays have the highest precision, medium delays are fairly precise, and information about short delays can have low granularity. On the other hand, when loss aversion is dominant (when β is low in Figure 6), both short and long delays have high precision.

To summarize the results of these experiments: When customers are risk conscious, information about longer delays should be more precise. In addition, the optimal precision for loss averse customers is robust to the constraint that the structure be ordinal: when customers are loss averse, information about the tails need to be precise whether the structure is onion or ordinal.

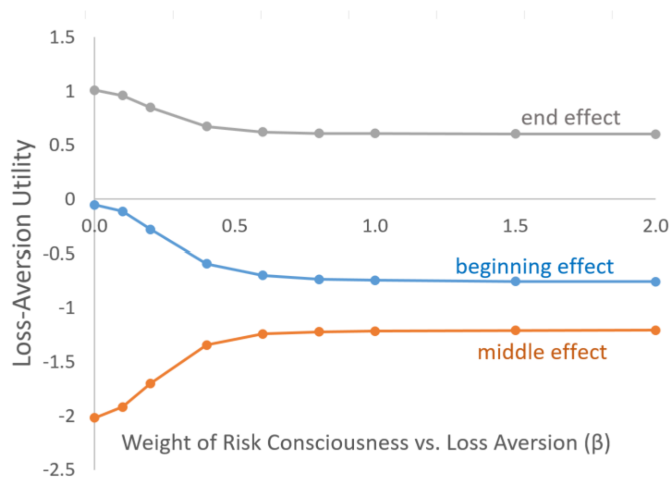


Figure 7 **Average beginning, middle and end effects on total loss-averse utility under optimal structures for randomly-generated priors with $\bar{\omega} = 7$.**

5.4. Timing of Utility: Beginning, Middle and End Effects

In Section 4.2, we used a continuous-time model to show that when $\beta = 0$ and there is a single announcement at $t = 0$, for a given class of priors the optimal information structure is either ordinal or onion. We also saw that the loss-averse disutility could be broken into beginning, middle, and end effects and found that under an onion structure the beginning effect is 0. That is, given an announcement at $t = 0$, the posterior mean remains equal to v^{-1}. To see if this is true for any arbitrary discrete prior, we plot the average beginning, middle and end effects under the optimal information structure for the 1,000 randomly-generated priors. Figure 7 shows that, consistent with the results of Proposition 4, when $\beta = 0$ the beginning effect is nearly zero.

Figure 7 also shows these effects as β rises. Note that for large beta, the risk-conscious customers are not directly affected by changes in the expected delay. Examining the timing of those changes, however, provides insight into optimal information design for all customers. We see in the Figure that as customers become more risk conscious, the middle effect—the disutility due to the passage of time— declines. That is, structures designed to reduce uncertainty can also reduce loss-aversion while waiting. Finally, the end effect is always positive because customers' expectations for delay are always greater than or equal to 0 just before they enter service. As β rises, however, the size of this positive surprise declines slightly under the optimal information structure. That is, reductions in uncertainty can reduce the pleasure from an unexpected end to the wait.

To summarize the results of these experiments: the main trade-off for a single, initial delay announcement is between the beginning effect and the passage of time. For loss-averse customers, it is optimal for the service provider to use an information structure that minimizes the beginning effect at the expense of a diminished experience as time passes. For risk conscious customers, the optimal ordinal structure reduces the impact of the passage of time, with the cost being more disappointment at the beginning.

6. Multiple Announcements

Rather than a single announcement at time $t = 0$, we now allow the service provider to make announcements in every period $0 \le t < \bar{\omega} - 1$. Each announced forecast is generated from an outcome, $m \in \mathcal{M}$ via $f^t_\omega(m)$, where $f^t_\omega(m) = 1$ if m is generated in period t for a customer starting service at time ω and $f^t_\omega(m) = 0$ otherwise. As before, the realized *sequence* of updated service start-time distributions is represented by $\tilde{\boldsymbol{g}}_\omega$ and is calculated via Bayes' rule,

$$g^t_T(m) = \frac{g^{t-1}_t f^t_T(m)}{\sum_{\tau=t+1}^{\bar{\omega}} g^{t-1}_\tau f^t_\tau(m)}, \text{ for } 1 \le t \le \bar{\omega} - 1,\ t+1 \le T \le \bar{\omega} \text{ and } m \in \mathcal{M}.$$

The customer's utility depends on that sequence, as in equation (6). Figure 8 shows an example of onion structures to generate multiple announcements. Note that some outcomes are labelled "arbitrary" in this figure. By the time those outcomes are generated, the delays ω for those customers have been fully revealed (posterior probabilities is equal to 1) and therefore any additional information has no effect. This will become clearer as we work through this example in Section 6.1.

As for the single-announcement case, the general model with multiple announcements is not tractable for analysis. Therefore, we consider two special cases. In Section 6.2 we will focus on a model with $\bar{\omega} = 3$, and in Section 6.3 we consider any $\bar{\omega}$, a symmetric prior distribution, and loss-averse customers. In Section 6.4 we describe additional numerical experiments with $\bar{\omega} = 7$ and announcements in every period.

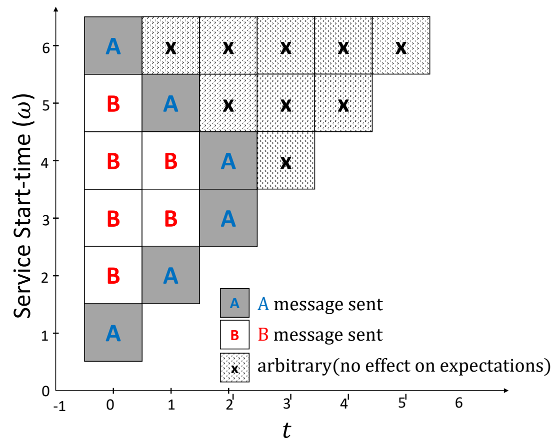


Figure 8 **Onion structures to generate multiple forecast announcements**

6.1. Numerical Example with Multiple Onion Structures

The following numerical example will show how multiple announcements shape customer expectations over time. In addition, this particular example demonstrates how onion structures can maximize the utility of loss averse customers. Given the information structure shown in Figure 8, we first calculate the posterior probabilities, expectations, and total loss aversion $U^L(\tilde{\boldsymbol{g}}_\omega)$. Then we consider the evolution of standard deviation over time and calculate utility due to risk consciousness.

As in Section 3.3, assume that customers have a symmetric prior service start-time distribution $\boldsymbol{g}^{-1} = (1/12, 2/12, 3/12, 3/12, 2/12, 1/12)$. Figure 9(a) shows how the information structure of Figure 8 affects the posterior expected waiting-times. All customers arrive with expectation $v^{-1} = 3.5$. Given the outcomes at $t = 0$, there is no change in expectations: Those who generate outcome A expect $\tilde{v}^0 = (1 \times 1/12 + 6 \times 1/12/(1/12 + 1/12) = 3.5$ and those who generate outcome B also have a posterior mean $\tilde{v}^0 = 3.5$. At time $t = 1$, customers with $\omega = 1$ go into service. This implies that customers who generate outcome A at $t = 0$ and do *not* go into service at $t = 1$ must have $\omega = 6$, which is the only other delay in the set for those who generate A at time $t = 0$. Therefore in Figure 9 we mark customers with $\omega = 1$ and $\omega = 6$ as fully revealed. Note that all outcomes for customers with $\omega = 6$ after $t = 0$ do not influence those customers' expectations and are therefore arbitrary.

At $t = 1$, customers with a start time $\omega \in \{2, 5\}$ generate outcome A and the remaining customers generate outcome B. Again, both sets of customers have posterior expected delays equal to their

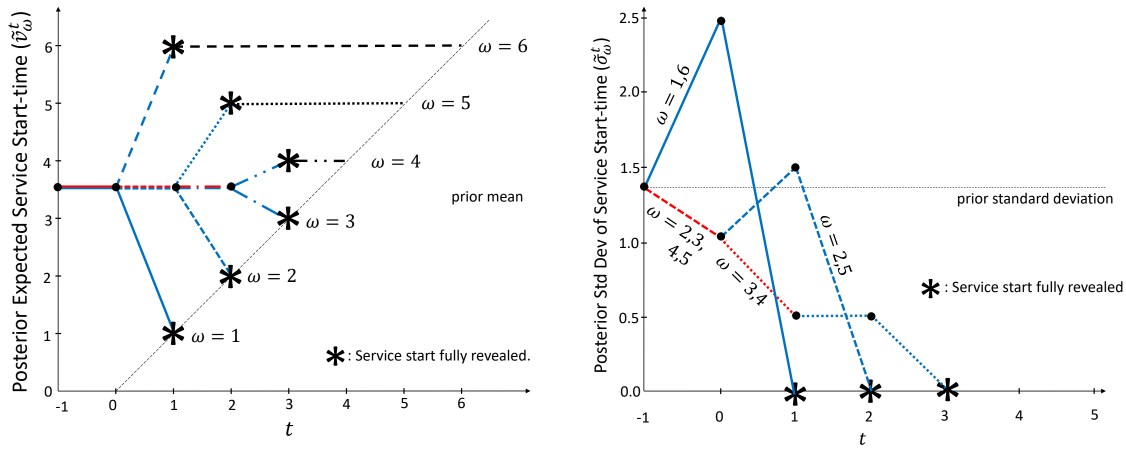


Figure 9 **(a) Expectation paths of $V^t(\tilde{g}^t)$. Blue paths are for customers whose most recent outcome was A and red paths are for customers whose most recent outcome was B (left); (b) Paths of posterior standard deviation $\sigma^t(\tilde{g}^t)$. Blue paths are for customers whose most recent outcome was A and red paths are for customers whose most recent outcome was B (right).**

priors, 3.5. At $t=2$, both delays $\omega=2$ and $\omega=5$ are revealed. Finally, at $t=2$, for customers with $\omega\in\{3,4\}$, their expected delay remains 3.5, and their delays are revealed at $t=3$.

To calculate the total utility from loss aversion for each customer, we again examine changes in the posterior means. For this information structure, we can see from Figure 9 that the utility of each customer is equal to the utility of the difference between the prior mean and the service time ω for that customer. Essentially, the onion structure keeps the posterior mean values flat until the true delay is revealed. As we will see below, this can dramatically reduce loss aversion penalties, as compared to ordinal and other information structures.

Now consider the evolution of the standard deviation (SD) shown in Figure 9. All customers begin with SD = 1.4. At $t=0$, customers with $\omega\in\{1,6\}$ who generate outcome A see a large increase in SD, to 2.5, but their SD drops to 0 at $t=1$. At $t=1$, customers with $\omega\in\{2,5\}$ generate outcome A, which again raises their SD, while the remaining customers $\omega\in\{3,4\}$ see a reduction in uncertainty.

Using the terms of the objective function defined in equation (6), we examine the overall performance of this multi-announcement strategy. Recall from Section 3.3 that for the single, ordinal structure $AAABBB$, the expected utility due to loss aversion is -1.02. For the multi-announcement, onion structure in Figure 8 the overall expected utility is -0.583, a significant improvement. As in Figure 9 (a), the onion structure holds the expected delay constant until service times are revealed, while the ordinal structure can elicit hope for a quick service and then disappointment, see e.g., the expectation paths of $\omega=2$ and $\omega=5$ in Figure 2 (a).

Now consider risk consciousness. For our single, ordinal structure $AAABBB$, the utility is -2.185, while for the multiple-announcement, onion structure in Figure 8, the utility is -2.27, a decline in utility despite having more forecast updates. The ordinal structure narrows the range of delays for all customers, reducing uncertainty, while the onion structure increases uncertainty by focusing information on the tails of the delay distribution. Therefore, we see that these multiple announcements with onion structures increase the utility of loss averse customers, as compared to a single onion announcement, but remain inferior to a single ordinal announcement for risk conscious customers.

6.2. Analysis of Multiple Announcements for $\bar{\omega} = 3$

Assume $\bar{\omega} = 3$, and we will consider the optimal information structure when there are announcements at both $t = 0$ and $t = 1$ (announcements at $t = 2$ and $t = 3$ are uninformative because all service start times are fully revealed by $t = 2$). For loss averse customers, the optimal information structure remains ABA, as in Section 4.1. With this structure, all wait times are fully revealed at $\omega = 1$: customers with $\omega = 1$ go into service, those with $\omega = 2$ have already received a unique forecast at $t = 0$, and those with $\omega = 3$ know that they are not $\omega = 1$. Therefore, the additional announcement at $t = 1$ is uninformative. The following Proposition follows directly from this reasoning:

Proposition 5. *Assume that there are announcements in every period for $\bar{\omega} = 3$. When the number of outcomes $M = 2$, and $\beta = 0$ (customers are loss averse), the information structure ABA is optimal at time $t = 0$ for any prior distribution and any $\lambda > 1$.*

In the next Section we will see that for loss averse customers the onion structure remains optimal over multiple periods, for any value of $\bar{\omega}$ and for a symmetric prior.

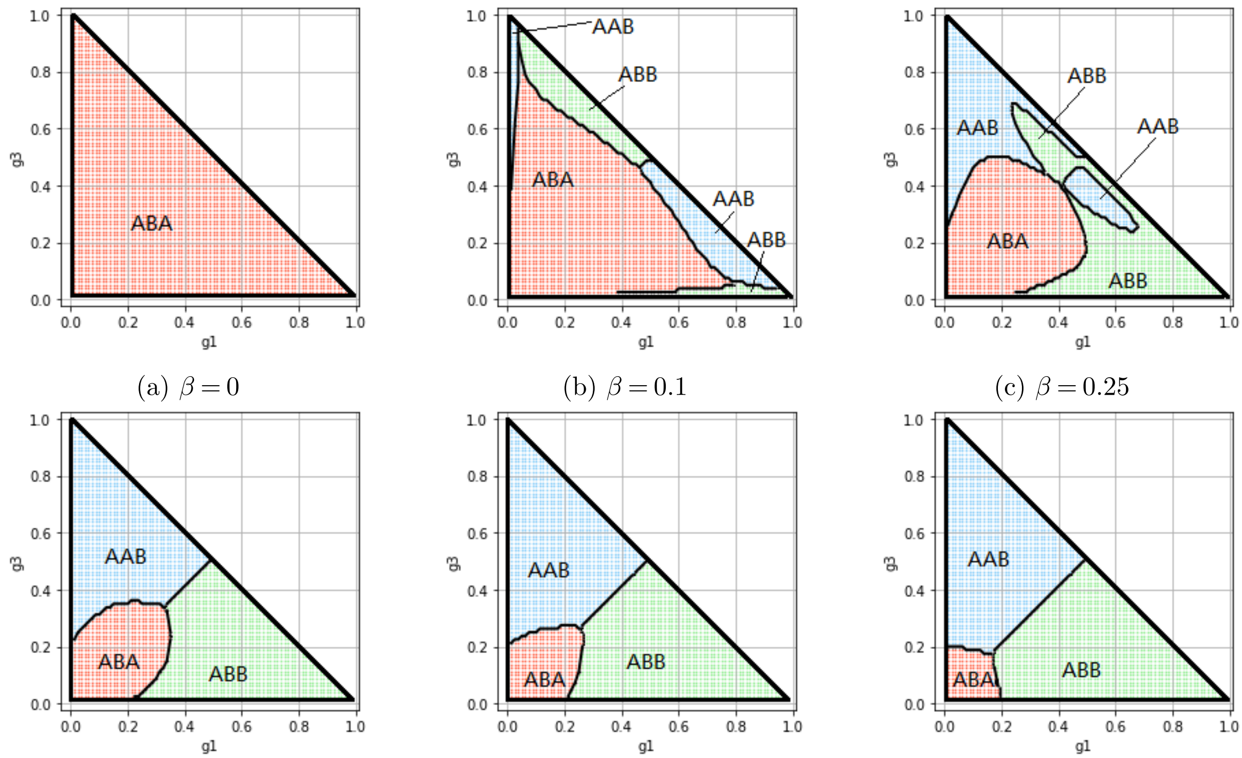
For risk conscious customers, again any of the three structures at $t = 0$ may be optimal. When there are multiple announcements, however, the boundaries differ from when there is a single announcement.

Proposition 6. *Assume that there are announcements in every period for $\bar{\omega} = 3$. When the number of outcomes $M = 2$, and $\beta = \infty$ (customers are risk conscious), with prior $\boldsymbol{g}^{-1} = (g_1^{-1}, 1 - g_1^{-1} - g_3^{-1}, g_3^{-1})$, then the optimal information structure at $t = 0$ is:*

- *AAB when $g_1^{-1} \leq 4g_3^{-1}$ and $g_1^{-1} + 5g_3^{-1} \geq 1$*
- *ABA when $g_1^{-1} + 5g_3^{-1} \leq 1$ and $2g_1^{-1} + g_3^{-1} \leq 1$*
- *ABB when $g_1^{-1} \geq 4g_3^{-1}$ and $2g_1^{-1} + g_3^{-1} \geq 1$*

Figure 10(f) shows optimality regions for this case. The area of optimality for the ABB structure is larger than in Figure 3(f), and is symmetric with the AAB structure. This is due to the fact that when there is a single announcement, customers with $\omega \in \{2, 3\}$ under the structure ABB in period $t = 0$ accrue disutility in period $t = 1$, before they learn their service start times in $t = 2$. An additional announcement in $t = 1$, however, reveals service start times to $\omega \in \{2, 3\}$, and therefore the ABB structure is equivalent to the AAB structure for symmetric prior distributions.

Figure 10 again shows the transition from the onion structure ABA to the (primarily) ordinal structures AAB and ABB as β increases. For intermediate values of β the pattern of optimal structures can be complex, and depends on the particular shape of the prior distribution.



(a) $\beta = 0$ (b) $\beta = 0.1$ (c) $\beta = 0.25$

(d) $\beta = 0.5$ (e) $\beta = 1$ (f) $\beta = \infty$

Figure 10 Optimal information structures for multiple announcements with $\bar{\omega} = 3$, as β varies

6.3. Analysis of Multiple Announcements with Symmetric Priors

We next analyze the optimization problem (1), assuming (i) the prior distribution g_ω^{-1} is discrete and symmetric, (ii) there is an announcement in each and every time-period, and (iii) customers are purely loss averse, $\beta = 0$. Given these assumptions, we find that an optimal information structure is the onion structure of Figure 8, which reveals extreme delays before revealing moderate delays. Given this information structure, the expectation paths follow what we call a *candelabra pattern*—a candelabra tipped on its side, as shown in Figure 9 (a).

First, we describe a general property of any optimal delay information structure for loss-averse customers.

Definition 1. For all $\omega \in \{1,\dots,\bar{\omega}\}$, the sequence of posterior expected service start-times $\{v^{-1}, \tilde{v}^0, \tilde{v}^1, \dots, \tilde{v}^{\omega-1}\}$ is a **monotonic** expectation path if

$$\forall \omega \le v^{-1} : \tilde{v}^0 \le v^{-1} \text{ and } \tilde{v}^t \le \tilde{v}^{t-1},\ t = 1,\dots,\omega$$
$$\forall \omega > v^{-1} : \tilde{v}^0 \ge v^{-1} \text{ and } \tilde{v}^t \ge \tilde{v}^{t-1},\ t = 1,\dots,\omega$$

That is, an expectation path is monotonic if it rises but never falls, or if it falls but never rises. Note that our definition of monotonicity is weak, in that a path may be flat and still be monotonic. All expectation paths in Figure 9 (a) are monotonic, but in Figure 2 (a), expectation paths for $\omega = 2, 4$ and 5 are not. Using Definition 1, we show in Lemma 1 that an information structure is optimal if every possible expectation path is monotonic, and we find an upper bound on expected customer utility.

Lemma 1. *Assume that there are announcements in every period. When the number of outcomes $M = 2$, and $\beta = 0$ (customers are loss averse), $\bar{U}^L = \mathbb{E}_\Omega\{u(v^{-1} - \Omega)\}$ is an upper bound on expected utility. A loss-averse customer's expected utility is equal to this upper bound if and only if every possible expectation path of that customer is monotonic.*

Note that $\bar{U}^L$ is the expected utility corresponding to the case where each customer is informed of her exact service start-time at time period $t = 0$. Now, we can present a result that characterizes the optimal multi-announcement information structure for loss-averse customers:

Proposition 7. *Assume that there are announcements in every period. When the number of outcomes $M = 2$, and $\beta = 0$ (customers are loss averse) and the prior distribution is symmetric, for any $\lambda > 1$ there exists an information structure, $f^t_\omega \in \{0,1\}$ for $\omega = 1, 2, \dots, \bar{\omega}$, $0 \le t < \omega$, with an onion structure that generates monotonic expectation paths and maximizes the expected utility.*

Proposition 7 is remarkable in the sense that for symmetric distributions, just two information outcomes ($M = 2$) in every period can achieve the same upper bound on utility as when there are $M = \bar{\omega}$ outcomes available for a single delay announcement. Essentially, the onion structure and resulting candelabra pattern for the expectation paths mitigates the impact of the passage of time by narrowing the service start-time distribution as time passes. For all customers the posterior expected service start-times remain equal to the initial expected start-times (v^{-1}) until the service start-time is fully revealed. According to Proposition 7, for symmetric distributions, it is optimal to inform perfectly, early on, customers with the latest start times ('very bad' news) or the earliest start times ('very good news'). As time progresses, customers with start times earlier than the latest or later than the earliest become progressively informed.

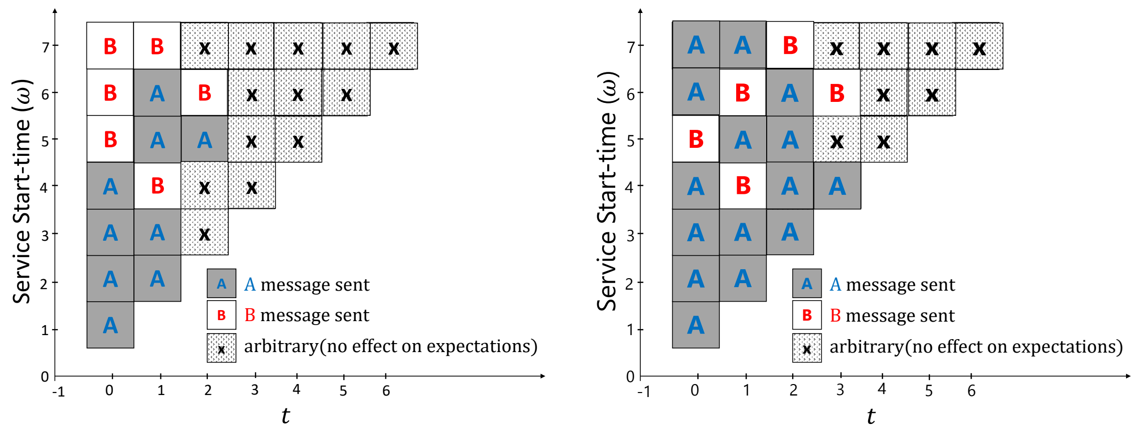


Figure 11 **(a) Structures for optimal multiple announcements with a uniform prior and $\beta = \infty$ (left); (b) Structures for optimal multiple announcements with a linearly increasing prior and $\beta = 0$ (right).**

6.4. Numerical Experiments for Multiple Announcements

In addition to the single-announcement numerical experiments of Section 5, we also conducted numerical experiments with announcements in every period. Figure 11 shows two examples of the optimal information structures. For the structures in Figure 11(a), customers are solely risk-conscious, and as in the single-announcement case, an ordinal structure is optimal at $t = 0$, with higher precision for the longer delays (a subset of size four for the shorter delays and size three for the longer delays). Subsequent optimal information structures, however, have complex (and probably unimplementable) patterns designed to leverage the information that has already been provided. The $AABAAB$ pattern in period 1, for example, reveals the identities of customers with $\omega = 4$ and $\omega = 7$; those two customers received different forecasts at $t = 0$ and are the only customers who generate 'B' at $t = 1$.

In Section 6.3 we saw that for solely risk-averse customers and for symmetric priors, onion structures are optimal in each and every period. Figure 11(b) shows that even for a non-symmetric prior, an onion structure can be optimal at $t = 0$. As in Figure 11(a), however, subsequent announcement structures are more complex and designed to strategically reveal information, given previous forecasts. Additional numerical experiments with other priors produce similar results at $t = 0$ (often onion structures for $\beta = 0$ and ordinal for $\beta = \infty$) and similarly complex structures in subsequent periods.

7. Summary and Implications for Practice

Providing delay information to customers in service systems is critical for managing customer satisfaction. Fuelled by the increased availability of data, improved data analysis techniques (such as machine learning) and improved communication technology, service firms have improved their forecasting systems to predict and announce customer delays. An improved forecast that reduces customer delay uncertainty, however, does not necessarily increase customer satisfaction with waiting. No forecasting system is perfect and customers may respond negatively to both uncertainty and disappointment. Therefore, we develop models to optimize delay information, given that customer utility is influenced by loss aversion and/or risk consciousness.

	customers are loss averse	**customers are risk conscious**
optimal structure	onion or customized to prior distribution	almost always ordinal
optimal precision	most precise in the tails	increasing precision for longer delays
utility timing	reduce beginning effect	reduce middle and end effect

Figure 12 Principles for optimal delay information design

Figure 12 summarizes the principles for optimal delay information design suggested by the analysis and numerical experiments. In the middle row of Figure 12, "precision," as defined in Section 5.3, refers to the size of the optimal delay partitions. For example when customers are primarily loss averse, information about both shorter and longer delays should be grouped into small subsets, while information about moderate delays may be less granular. In the bottom row, "utility timing" specifies the optimal timing for information that leads to significant revisions in customer expectations, where changes in expectations can have a significant positive or negative impact on utility. When customers are loss averse, the initial delay forecast should be designed to hold expectations steady, so that delay information is revealed during the wait and when service begins. When customers are risk conscious, it is optimal to 'deliver the bad news up-front' by providing forecasts that accurately reveal long vs. short delays. Note that our underlying model does not give greater weight to experiences at any particular time during the wait, as would be suggested by behavioral economics, e.g., Redelmeier et al. (2003). Therefore, the prescriptions on the bottom row of Figure 12 are generated, via optimization, from the assumptions of loss aversion and risk consciousness. Additional research may add explicit beginning, peak, and/or end-effects to the model primitives.

In general, our modeling framework makes few assumptions, and by identifying optimal information structures over a large space of possible structures, we generate upper bounds on customer utility. In addition, the optimal structures provide guidance on how to achieve those upper bounds, although practical implementation may be constrained by data needs and customer acceptance.

In general the cost of waiting might be divided into two components: (i) the emotional cost due to anxiety, impatience and disappointment and (ii) the opportunity cost of the time spent waiting. In our model, the objective of the firm is to provide information so as to optimize the first, emotional component. In the queueing literature, this cost is often assumed to have a particular form (e.g., convex increasing), and our model provides a micro-foundation for understanding this cost.

Our objective does not explicitly include the customer's opportunity cost for the time spent waiting. Opportunity cost could be incorporated in our model, but as long as the waiting-time experience does not affect the duration of delay, inclusion of opportunity cost will not change the optimal information structure. Yu et al. (2021), however, point out that changes in customers' beliefs and experiences can change waiting-times, via reneging behavior (freeing up capacity for the remaining customers) and by directly influencing the customers' service times (unhappy customers may demand longer service). To examine these effects, our model might be embedded into a model of a queueing system.

Recall that our model allows the forecasting technology to perfectly distinguish between adjacent delays, i.e. it is possible for $|f_\omega(m) - \mathrm{f}_{\omega-1}(m)| = 1$. Further research would incorporate forecasting technologies that are less discriminating. An alternate model would add a constraint $|f_\omega(m) - \mathrm{f}_{\omega-1}(m)| \leq \epsilon < 1$ and allow for overlapping sets. For example, consider two outcomes $(M = 2)$, $\bar{\omega} = 6$, and $\epsilon = 0.5$. Consider a structure with outcome A for $\omega \in \{1, 2, 3, 4\}$ and B for $\omega \in \{3, 4, 5, 6\}$, and with $f_1^0 = f_2^0 = 1$, $f_3^0 = f_4^0 = 0.5$, and $f_5^0 = f_6^0 = 0$. Therefore, customers who generate outcome A (B) would have a lower (higher) posterior mean, but both would in essence have '100% confidence intervals' around the updated means: 1 to 4 for customers who generate A and 3 to 6 for those who generate B.

Our modeling framework may also guide developers of forecasting systems. Statistical methods predicting wait times typically minimize the MSE. Our framework suggests that the customer experience is most sensitive to the forecast accuracy of long wait times (for risk conscious customers) and also short wait times (for loss averse customers). This provides a theoretical foundation for non-symmetric MSE criteria, such as in Bassamboo and Ibrahim (2021), Section EC.6. In general, our paper suggests that a new, fruitful area of research may focus on how customer wait-time forecasting and messaging systems can be designed to optimize the customer experience.

References

Aflaki, Sam, Ioana Popescu. 2013. Managing retention in service relationships. *Management Science* **60**(2) 415–433.

Akşin, Zeynep, Baris Ata, Seyed Morteza Emadi, Che-Lin Su. 2016. Impact of delay announcements in call centers: An empirical approach. *Operations Research* **65**(1) 242–265.

Allon, Gad, A Bassamboo. 2011. The impact of delaying the delay announcements. *Management Science* **59**(5) 1198–1210.

Allon, Gad, Achal Bassamboo, Itai Gurvich. 2011. "we will be right with you": Managing customer expectations with vague promises and cheap talk. *Operations research* **59**(6) 1382–1394.

Ang, Erjie, Sara Kwasnick, Mohsen Bayati, Erica L Plambeck, Michael Aratow. 2016. Accurate emergency department wait time prediction. *Manufaturing and Service Operations Management* .

Ansari, Sina, Laurens Debo, Seyed Iravani, Sanjeev Malik. 2018. Engineering the delay announcement to improve patient satisfaction. Tech. rep., Working Paper.

Anunrojwong, Jerry, Krishnamurthy Iyer, David Lingenbrink. 2020. Persuading risk-conscious agents: A geometric approach. *Available at SSRN 3386273* .

Anunrojwong, Jerry, Krishnamurthy Iyer, Vahideh Manshadi. 2022. Information design for congested social services: Optimal need-based persuasion. *Management Science* .

Armony, Mor, Nahum Shimkin, Ward Whitt. 2009. The impact of delay announcements in many-server queues with abandonment. *Operations Research* **57**(1) 66–81.

Bassamboo, Achal, Rouba Ibrahim. 2021. A general framework to compare announcement accuracy: Static vs. les-based announcement. *Management Science* **67**(7) 4191–4208.

Cui, S, S Veeraraghavan. 2016. Blind queues: The impact of consumer beliefs on revenues and congestion. *Management Science* **62**(12) 3656–3672.

Das Gupta, Aparupa, Uday S Karmarkar, Guillaume Roels. 2015. The design of experiential services with acclimation and memory decay: Optimal sequence and duration. *Management Science* **62**(5) 1278–1296.

Dong, Jing, Elad Yom-Tov, Galit B Yom-Tov. 2015. The impact of delay announcements on hospital network coordination and waiting times. Tech. rep., Working Paper.

Ely, Jeffrey, Alexander Frankel, Emir Kamenica. 2015. Suspense and surprise. *Journal of Political Economy* **123**(1) 215–260.

Gaur, Vishal, Young-Hoon Park. 2007. Asymmetric consumer learning and inventory competition. *Management Science* **53**(2) 227–240.

Gehlbach, Scott, Konstantin Sonin. 2014. Government control of the media. *Journal of Public Economics* **118** 163–171.

Gick, Wolfgang, Thilo Pausch. 2012. Persuasion by stress testing: Optimal disclosure of supervisory information in the banking sector .

Goldstein, Itay, Yaron Leitner. 2018. Stress tests and information disclosure. *Journal of Economic Theory* **177** 34–69.

Groeger, Lena. 2019. Er inspector by propublica. https://projects.propublica.org/emergency/.

Guda, Harish, Milind Dawande, Ganesh Janakiraman. 2020. The economics of process transparency. *Available at SSRN 3715037* .

Guo, Pengfei, Paul Zipkin. 2007. Analysis and comparison of queues with different levels of delay information. *Management Science* **53**(6) 962–970.

Hassin, Refael. 1986. Consumer information in markets with random product quality: The case of queues and balking. *Econometrica: Journal of the Econometric Society* 1185–1195.

Hassin, Refael. 2007. Information and uncertainty in a queuing system. *Probability in the Engineering and Informational Sciences* **21**(03) 361–380.

Health, Vancouver Coastal. 2012. Vancouver, richmond and north shore emergency department wait times. http://www.edwaittimes.ca/WaitTimes.aspx.

Hu, Ming, Yang Li, Jianfu Wang. 2017. Efficient ignorance: Information heterogeneity in a queue. *Management Science* .

Hui, Michael K, David K Tse. 1996. What to tell consumers in waits of different lengths: An integrative model of service evaluation. *The Journal of Marketing* 81–90.

Jehiel, Philippe. 2014. On transparency in organizations. *The Review of Economic Studies* **82**(2) 736–761.

Kahneman, Daniel, Amos Tversky. 1979. Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society* 263–291.

Kahneman, Daniel, Peter P Wakker, Rakesh Sarin. 1997. Back to bentham? explorations of experienced utility. *The quarterly journal of economics* **112**(2) 375–406.

Kamenica, Emir, Matthew Gentzkow. 2011. Bayesian persuasion. *American Economic Review* **101**(6) 2590–2615.

Kolotilin, Anton. 2015. Experimental design to persuade. *Games and Economic Behavior* **90** 215–226.

Kroll, Yoram, Haim Levy, Harry M. Markowitz. 1984. Mean-variance versus direct utility maximization. *The Journal of Finance* **34**(1) 47–61.

Kuo, Y. H., N. B. Chan, J. M. Leung, H. Meng, A. M. C. So, K. K. Tsoi, C. A. Graham. 2020. An integrated approach of machine learning and systems thinking for waiting time prediction in an emergency department. *International journal of medical informatics* **139**.

Larson, Richard C. 1988. There's more to a line than its wait. *Technology review* **91**(5) 60–67.

Lingenbrink, David, Krishnamurthy Iyer. 2019. Optimal signaling mechanisms in unobservable queues. *Operations research* **67**(5) 1397–1416.

Loewenstein, George, Andras Molnar. 2018. The renaissance of belief-based utility in economics. *Nature Human Behaviour* **2**(3) 166–167.

Long, Xiaoyang, Javad Nasiry. 2014. Prospect theory explains newsvendor behavior: The role of reference points. *Management Science* **61**(12) 3009–3012.

Maister, David H. 1985. The psychology of waiting lines. J Czepiel., M.R. Solomon, C.F. Suprenant, eds., *The Service Encounter: Managing Employee/customer Interaction in Service Businesses*. Lexington Books.

Markowitz, Harry. 1952. Portfolio selection. *The Journal of Finance* **7**(1) 77–91.

Mogre, Riccardo, Tava Olsen, Valery Pavlov. 2020. Quoting lead times when quotes directly affect customer satisfaction. Tech. rep., University of Auckland Business School.

Polevoi, Steven K., James V. Quinn, Nathan R. Kramer. 2005. Factors associated with patients who leave without being seen. *Academic Emergency Medicine* **12**(3) 232–236.

Rayo, Luis, Ilya Segal. 2010. Optimal information disclosure. *Journal of political Economy* **118**(5) 949–987.

Redelmeier, Donald A., Joel Katz, Daniel Kahneman. 2003. Memories of colonoscopy: a randomized trial. *Pain* **104**(1) 187–194.

Schweizer, Nikolaus, Nora Szech. 2018. Optimal revelation of life-changing information. *Management Science* **64**(11) 5250–5262.

Shah, Shital, Anay Patel, Dino P Rumoro, Samuel Hohmann, Francis Fullam. 2015. Managing patient expectations at emergency department triage. *Patient Experience Journal* **2**(2) 31–44.

Shumsky, Robert A. 1998. Optimal updating of forecasts for the timing of future events. *Management science* **44**(3) 321–335.

Tversky, Amos, Daniel Kahneman. 1991. Loss aversion in riskless choice: A reference-dependent model. *The quarterly journal of economics* **106**(4) 1039–1061.

Tversky, Amos, Daniel Kahneman. 1992. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty* **5**(4) 297–323.

Whitt, Ward. 1999. Improving service by informing customers about anticipated delays. *Management science* **45**(2) 192–207.

Xie, Bin, Sabrina Youash. 2011. The effects of publishing emergency department wait time on patient utilization patterns in a community with two emergency department sites: a retrospective, quasi-experiment design. *International journal of emergency medicine* **4**(1) 29.

Yu, Qiuping. 2020. When providing wait times, it pays to underpromise and overdeliver. *Harvard Business Review Online* .

Yu, Qiuping, Gad Allon, Achal Bassamboo. 2016. How do delay announcements shape customer behavior? an empirical study. *Management Science* **63**(1) 1–20.

Yu, Qiuping, Gad Allon, Achal Bassamboo. 2021. The reference effect of delay announcements: A field experiment. *Management Science* **67**(12) 7417–7437.

Yu, Qiuping, Gad Allon, Achal Bassamboo, Seyed Iravani. 2017. Managing customer expectations and priorities in service systems. *Management Science* **64**(8) 3942–3970.

Yu, Qiuping, Yiming Zhang, Yong-Pin Zhou. 2020. Delay information in virtual queues: A large-scale field experiment on a ride-sharing platform. *Working Paper. Available at SSRN: https://ssrn.com/abstract=3687302 or http://dx.doi.org/10.2139/ssrn.3687302* .

Appendix

1. Proofs of Statements

LEMMA 2. *Given that customers are only loss averse ($\beta = 0$), the total expected utility across all customers can be expressed as,*

$$(\lambda - 1)\mathbb{E}_\Omega[\sum_{t=0}^{\Omega-1}\{\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^{\boldsymbol{t-1}}) - \tilde{v}^t(\tilde{\boldsymbol{g}}^{\boldsymbol{t}})\}^+ + \{\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^{\boldsymbol{t-1}}) - \Omega\}^+] \tag{10}$$

Proof of Lemma 2 For convenience we will not make explicit the dependence of $\tilde{v}^t$ on $\tilde{g}^t$, i.e., $\tilde{v}^t(\tilde{\boldsymbol{g}}^{\boldsymbol{t}}) = \tilde{v}^t$. The expected utility is $\mathbb{E}_\Omega[U_\Omega^L]$, where

$$U_\omega^L = \sum_{t=0}^{\omega-1}(\lambda\{\tilde{v}^{t-1} - \tilde{v}^t\}^+ + \{\tilde{v}^{t-1} - \tilde{v}^t\}^-) + \lambda\{\tilde{v}^{t-1} - \omega\}^+ + \{\tilde{v}^{\omega-1} - \omega\}^-$$

and with the martingale property, we have that $\mathbb{E}_\Omega[\tilde{v}^{t-1}] = \mathbb{E}_\Omega[\tilde{v}^t] = v^{-1}$. Furthermore,

$$\lambda\{x\}^+ + \{x\}^- = \lambda\{x\}^+ + \underbrace{\{x\}^- - x}_{=\min\{0,x\}-x=-\max\{x,0\}} + x = (\lambda - 1)\{x\}^+ + x$$

And thus with $x = \tilde{v}^{t-1} - \tilde{v}^t$, or $x = \tilde{v}^{t-1} - \omega$, we have that $\mathbb{E}_\Omega[\lambda\{x\}^+ + \{x\}^-] = \mathbb{E}_\Omega[(\lambda-1)\{x\}^+ + x] = (\lambda - 1)\mathbb{E}_\Omega[\{x\}^+] + \mathbb{E}_\Omega[x] = (\lambda - 1)\mathbb{E}_\Omega[\{x\}^+]$, from which follows that

$$\begin{aligned}
\mathbb{E}_\Omega[U_\Omega^L] &= \mathbb{E}_\Omega[\sum_{t=0}^{\Omega-1}(\lambda\{\tilde{v}^{t-1} - \tilde{v}^t\}^+ + \{\tilde{v}^{t-1} - \tilde{v}^t\}^-) + \lambda\{\tilde{v}^{t-1} - \Omega\}^+ + \{\tilde{v}^{\Omega-1} - \Omega\}^-] \\
&= \mathbb{E}_\Omega[\sum_{t=0}^{\Omega-1}(\lambda - 1)\{\tilde{v}^{t-1} - \tilde{v}^t\}^+ + (\lambda - 1)\{\tilde{v}^{t-1} - \Omega\}^+] \\
&= (\lambda - 1)\mathbb{E}_\Omega[\sum_{t=0}^{\Omega-1}\{\tilde{v}^{t-1} - \tilde{v}^t\}^+ + \{\tilde{v}^{t-1} - \Omega\}^+]
\end{aligned}$$

□

Proof of Proposition 1 To derive algebraic expressions for the utility due to loss aversion with $\bar{\omega} = 3$, we first reformulate the objective.

With Lemma 2, for $\lambda > 1$ the optimal structure does not depend on the specific value of λ. In addition, the objective can be written as the sum of all *positive changes* in expected delays.

Now for $\bar{\omega} = 3$, we write out explicit expressions for the total loss-aversion disutility in terms of the prior probabilities g_1, g_2, and g_3. (We will express the objective as minimizing a positive amount of disutility rather than maximizing a negative utility.) First we express the prior expectation as $v^{-1} = g_1 + 2g_2 + 3(1 - g_1 - g_2) = 3 - 2g_1 - g_2$.

When the structure is ABB, the sum of positive changes in expectations is:

$$U_1 = 0 + g_2(\frac{2g_2 + 3(1-g_1-g_2)}{1-g_1} - (3-2g_1-g_2)) + (1-g_1-g_2)(3-(3-2g_1-g_2))$$

When the structure is AAB, the sum of positive changes in expectation is:

$$U_2 = 0 + g_2(2 - \frac{g_1+2g_2}{g_1+g_2}) + (1-g_1-g_2)(3-(3-2g_1-g_2))$$

When the structure is ABA, the sum of positive changes in expectation depends on the prior expectation: When the prior expectation is greater than 2, i.e., $3-2g_1-g_2 \geq 2$, the sum of positive changes in expectation is:

$$U_3^{2+} = g_1(\frac{g_1 + 3(1-g_1-g_2)}{1-g_2} - (3-2g_1-g_2)) + (1-g_1-g_2)(3-(3-2g_1-g_2))$$

When the prior expectation is less than 2, i.e., $3-2g_1-g_2 \leq 2$, the sum of positive changes in expectation is:

$$U_3^{2-} = g_2(2-(3-2g_1-g_2)) + (1-g_1-g_2)(3-\frac{g_1+3(1-g_1-g_2)}{1-g_2})$$

Now we describe the relationship between U_1 and U_2:

$$\begin{aligned}
&U_1 \geq U_2 \\
&\iff g_2(\frac{2g_2 + 3(1-g_1-g_2)}{1-g_1} - (3-2g_1-g_2)) + (1-g_1-g_2)(3-(3-2g_1-g_2)) \\
&\geq g_2(2-\frac{g_1+2g_2}{g_1+g_2}) + (1-g_1-g_2)(3-(3-2g_1-g_2)) \\
&\iff \frac{3-3g_1-g_2}{1-g_1} - (3-2g_1-g_2) \geq 2 - \frac{g_1+2g_2}{g_1+g_2} \\
&\iff 2g_1 + g_2 - \frac{g_2}{1-g_1} \geq \frac{g_1}{g_1+g_2} \\
&\iff \frac{g_1(1-g_1+g_2)(2g_1+g_2-1)}{(1-g_1)(g_1+g_2)} \geq 0 \\
&\iff 2g_1 + g_2 \geq 1
\end{aligned}$$

This inequality is equivalent to having the prior expectation less than 2 ($3-2g_1-g_2 \leq 2$). Thus, when the prior expectation is less than 2, $U_1 \geq U_2$, meaning the positive changes for structure AAB is larger than ABB, the latter structure will result in smaller disutility from loss-aversion, and vice versa.

Thus, we only need to show that $U_2 \geq U_3^{2-}$ and $U_1 \geq U_3^{2+}$ to prove the proposition. We first show $U_2 \geq U_3^{2-}$ when $2g_1 + g_2 \geq 1$:

$$
\begin{aligned}
&U_1 \geq U_3^{2+} \\
&g_2(\frac{2g_2 + 3(1-g_1-g_2)}{1-g_1} - (3-2g_1-g_2)) + (1-g_1-g_2)(3-(3-2g_1-g_2)) \\
&\geq g_2(2-(3-2g_1-g_2)) + (1-g_1-g_2)(3 - \frac{g_1 + 3(1-g_1-g_2)}{1-g_2}) \\
&\Longleftrightarrow \frac{g_1 g_2}{g_1+g_2} + (1-g_1-g_2)(2g_1+g_2) \geq g_2(2g_1+g_2-1) + \frac{2g_1(1-g_1-g_2)}{1-g_2} \\
&\Longleftrightarrow \frac{g_2(g_1+g_2-1)(2g_1^2+5g_1g_2-3g_1+2g_2^2-2g_2)}{(1-g_2)(g_1+g_2)} \geq 0 \\
&\Longleftrightarrow 2g_1^2+5g_1g_2-3g_1+2g_2^2-2g_2 \leq 0 \qquad (\text{since } g_1+g_2 \leq 1 \text{ and } g_2 \leq 1) \\
&\Longleftrightarrow g_1(2g_1+3g_2-3) + 2g_1(g_1+g_2-1) \leq 0
\end{aligned}
$$

Because $2g_1 + 3g_2 - 3$ and $g_1 + g_2 - 1$ are obviously negative, the inequality holds, we have $U_2 \geq U_3^{2-}$.

Now we show $U_1 \geq U_3^{2+}$ when $2g_1 + g_2 \leq 1$:

$$
\begin{aligned}
&U_2 \geq U_3^{2-} \\
&\Longleftrightarrow g_2(2 - \frac{g_1+2g_2}{g_1+g_2}) + (1-g_1-g_2)(3-(3-2g_1-g_2)) \\
&\geq g_1(\frac{g_1+3(1-g_1-g_2)}{1-g_2} - (3-2g_1-g_2)) + (1-g_1-g_2)(3-(3-2g_1-g_2)) \\
&\Longleftrightarrow g_2(\frac{g_1}{g_1+g_2}) \geq g_1(2g_1+g_2 - \frac{2g_1}{1-g_2}) \\
&\Longleftrightarrow \frac{g_2}{g_1+g_2} + \frac{2g_1}{1-g_2} \geq 2g_1 + g_2
\end{aligned}
$$

Since $g_1 + g_2 \leq 1$, and $1 - g_2 \leq 1$, the inequality holds, $U_2 \geq U_3^{2-}$.

□

Proof of Proposition 2 We can first write out an expression for risk conscious disutility. Denote W_1, W_2 and W_3 as the expected risk conscious disutility of customers with structure AAB, ABA, and ABB, respectively. When the structure is AAB, we can write out the explicit expression for the cumulative standard deviation. Note that only customers with starting times 1 and 2 will have uncertainty:

$$
\begin{aligned}
W_1 &= (1-g_3)\sqrt{\frac{g_1}{1-g_3}(1-\frac{g_1+2(1-g_1-g_3)}{1-g_3})^2 + \frac{1-g_1-g_3}{1-g_3}(2-\frac{g_1+2(1-g_1-g_3)}{1-g_3})^2 + 0} \\
&= (1-g_3)\sqrt{\frac{g_1}{1-g_3}(\frac{1-g_1-g_3}{1-g_3})^2 + \frac{1-g_1-g_3}{1-g_3}(\frac{g_1}{1-g_3})^2} \\
&= \sqrt{g_1(1-g_1-g_3)}
\end{aligned}
$$

Similarly for structure ABA:

$$\begin{aligned}W_2 &= (g_1+g_3)\sqrt{\frac{g_1}{g_1+g_3}(1-\frac{g_1+3g_3}{g_1+g_3})^2+\frac{g_3}{g_1+g_3}(3-\frac{g_1+3g_3}{g_1+g_3})^2}+0\\&= (g_1+g_3)\sqrt{\frac{g_1}{g_1+g_3}(\frac{-2g_3}{g_1+g_3})^2+\frac{g_3}{g_1+g_3}(\frac{2g_1}{g_1+g_3})^2}\\&= 2\sqrt{g_1g_3}\end{aligned}$$

For structure ABB, the customers with starting time 2 and 3 will accumulate the same amount of uncertainty in $t=0$ and $t=1$:

$$\begin{aligned}W_3 &= 2(1-g_1)\sqrt{\frac{1-g_1-g_3}{1-g_1}(2-\frac{2(1-g_1-g_3)+3g_3}{1-g_1})^2+\frac{g_3}{1-g_1}(3-\frac{2(1-g_1-g_3)+3g_3}{1-g_1})^2}+0\\&= 2(1-g_1)\sqrt{\frac{1-g_1-g_3}{1-g_1}(\frac{g_3}{1-g_1})^2+\frac{g_3}{1-g_1}(\frac{1-g_1-g_3}{1-g_1})^2}\\&= 2\sqrt{g_3(1-g_1-g_3)}\end{aligned}$$

Thus, we find for W_1, W_2, and W_3,

$$\begin{aligned}&W_1 \geq W_2\\&\iff \sqrt{g_1(1-g_1-g_3)} \geq 2\sqrt{g_1g_3}\\&\iff 1-g_1-g_3 \geq 4g_3\\&\iff g_1+5g_3 \leq 1\end{aligned}$$

$$\begin{aligned}&W_2 \geq W_3\\&\iff 2\sqrt{g_1g_3} \geq 2\sqrt{g_3(1-g_1-g_3)}\\&\iff g_1 \geq 1-g_1-g_3\\&\iff 2g_1+g_3 \leq 1\end{aligned}$$

$$\begin{aligned}&W_1 \geq W_3\\&\iff \sqrt{g_1(1-g_1-g_3)} \geq 2\sqrt{g_3(1-g_1-g_3)}\\&\iff g_1 \geq 4g_3\end{aligned}$$

These inequalities form the boundaries of the outcomes.

□

Proof of Proposition 3 We rewrite the negative of the objective function

$$(\lambda-1)\left(\int_1^{\bar{\omega}}\sum_m[\{v_m^0-v^{-1}\}^+ + (v_m(\omega)-v^{-1})]f_m(\omega)g^{-1}(\omega)d\omega\right) \text{ with}$$
$$v_m(\omega)=\frac{\int_\omega^{\bar{\omega}}\tilde{\omega}f_m(\tilde{\omega})g^{-1}(\tilde{\omega})d\tilde{\omega}}{\int_\omega^{\bar{\omega}}f_m(\tilde{\omega})g^{-1}(\tilde{\omega})d\tilde{\omega}} \text{ and } v_m^0=v_m(1)$$

We define

$$L\triangleq\int_1^{\bar{\omega}}[\frac{\int_\omega^{\bar{\omega}}\tilde{\omega}f(\tilde{\omega})g^{-1}(\tilde{\omega})d\tilde{\omega}}{\int_\omega^{\bar{\omega}}f(\tilde{\omega})g^{-1}(\tilde{\omega})d\tilde{\omega}}f(\omega)+\frac{\int_\omega^{\bar{\omega}}\tilde{\omega}(1-f(\tilde{\omega}))g(\tilde{\omega})d\tilde{\omega}}{\int_\omega^{\bar{\omega}}(1-f(\tilde{\omega}))g(\tilde{\omega})d\tilde{\omega}}(1-f(\omega))]g^{-1}(\omega)\}d\omega$$

with $x(\omega)=f(\omega)g^{-1}(\omega)$ or $x(\omega)=(1-f(\omega))g^{-1}(\omega)$, we have the following structure:

$$L_X\triangleq\int_1^{\bar{\omega}}\frac{\int_\omega^{\bar{\omega}}\tilde{\omega}x(\tilde{\omega})d\tilde{\omega}}{\int_\omega^{\bar{\omega}}x(\tilde{\omega})d\tilde{\omega}}x(\omega)d\omega \text{ and } X(\omega)=\int_\omega^{\bar{\omega}}x(\tilde{\omega})d\tilde{\omega} \text{ and } X(\bar{\omega})=0$$

Then

$$\begin{aligned}
L_X &= -\int_1^{\bar{\omega}}\frac{\int_\omega^{\bar{\omega}}\tilde{\omega}dX(\tilde{\omega})}{X(\omega)}x(\omega)d\omega=\int_1^{\bar{\omega}}\frac{\tilde{\omega}X(\tilde{\omega})|_\omega^{\bar{\omega}}-\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{X(\omega)}dX(\omega) \text{ with } \tilde{\omega}X(\tilde{\omega})|_\omega^{\bar{\omega}}=-\omega X(\omega)\\
&= -\int_1^{\bar{\omega}}(\omega+\frac{\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{X(\omega)})dX(\omega)=-(\omega+\frac{\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{X(\omega)})X(\omega)\Big|_1^{\bar{\omega}}+\int_1^{\bar{\omega}}X(\omega)d(\omega+\frac{\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{X(\omega)})\\
&= (1+\frac{\int_1^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{X(1)})X(1)+\int_1^{\bar{\omega}}X(\omega)d(\omega+\frac{\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{X(\omega)})\\
&= X(1)+\int_1^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}+\int_1^{\bar{\omega}}X(\omega)d\omega+\int_1^{\bar{\omega}}X(\omega)\frac{-(X(\omega))^2+x(\omega)\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{(X(\omega))^2}d\omega\\
&= X(1)+\int_1^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}+\int_1^{\bar{\omega}}\frac{x(\omega)\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}}{X(\omega)}d\omega\\
&= X(1)+\int_1^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}-\int_1^{\bar{\omega}}\frac{d}{d\omega}\ln(X(\omega))\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}d\omega\\
&= X(1)+\int_1^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}-\int_1^{\bar{\omega}}\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}d\ln(X(\omega))\\
&= X(1)+\int_1^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}-\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}\ln(X(\omega))\Big|_1^{\bar{\omega}}+\int_1^{\bar{\omega}}\ln(X(\omega))d\int_\omega^{\bar{\omega}}X(\tilde{\omega})d\tilde{\omega}\\
&= X(1)+\int_1^{\bar{\omega}}X(\omega)d\omega+\ln(X(1))\int_1^{\bar{\omega}}X(\omega)d\omega-\int_1^{\bar{\omega}}X(\omega)\ln(X(\omega))d\omega\\
&= X(1)+\int_1^{\bar{\omega}}\{(1+\ln(X(1)))X(\omega)-X(\omega)\ln(X(\omega))\}d\omega\\
&= X(1)+\int_1^{\bar{\omega}}X(\omega)\left\{1-\ln(\frac{X(\omega)}{X(1)})\right\}d\omega
\end{aligned}$$

Furthermore let

$$v(X)\triangleq\frac{\int_1^{\bar{\omega}}\tilde{\omega}x(\tilde{\omega})d\tilde{\omega}}{\int_1^{\bar{\omega}}x(\tilde{\omega})d\tilde{\omega}}=1+\frac{\int_1^{\bar{\omega}}X(\omega)d\omega}{X(1)}$$

Note: from now on, we use the tail distributions for $\bar{G}$ and $\bar{F}$; $\bar{G}(\omega)=\int_{\omega}^{\bar{\omega}} g(\omega)d\omega$ and $\bar{F}(\omega)=\int_{\omega}^{\bar{\omega}} f(\omega)d\omega$. Thus:

$$L=\bar{F}(1)+\int_1^{\bar{\omega}} \bar{F}(\omega)\left\{1-\ln(\frac{\bar{F}(\omega)}{\bar{F}(1)})\right\}d\omega+\bar{G}(1)-\bar{F}(1)+\int_1^{\bar{\omega}}(\bar{G}(\omega)-\bar{F}(\omega))\left\{1-\ln(\frac{\bar{G}(\omega)-\bar{F}(\omega)}{\bar{G}(1)-\bar{F}(1)})\right\}d\omega$$
$$=1+\int_1^{\bar{\omega}} \bar{G}(\omega)d\omega-\int_1^{\bar{\omega}} \bar{F}(\omega)\ln(\frac{\bar{F}(\omega)}{\bar{F}(1)})d\omega-\int_1^{\bar{\omega}}(\bar{G}(\omega)-\bar{F}(\omega))\ln(\frac{\bar{G}(\omega)-\bar{F}(\omega)}{1-\bar{F}(1)})d\omega$$

and

$$I(\bar{F})=v^{-1}-\int_1^{\bar{\omega}} \ln(\frac{\bar{F}(\omega)}{\bar{F}(1)})^{\bar{F}(\omega)}d\omega-\int_1^{\bar{\omega}} \ln(\frac{\bar{G}(\omega)-\bar{F}(\omega)}{1-\bar{F}(1)})^{(\bar{G}(\omega)-\bar{F}(\omega))}d\omega$$

Now,

$$G_A(\omega)=\frac{\bar{F}(\omega)}{\bar{F}(1)} \text{ and } G_B(\omega)=\frac{\bar{G}(\omega)-\bar{F}(\omega)}{1-\bar{F}(1)} \text{ and } p_A=\bar{F}(1) \text{ and } p_B=1-\bar{F}(1)$$

then with $T(G)\triangleq -G\ln(G)$, we have

$$L=\int_1^{\bar{\omega}}\left\{p_A T(\bar{G}_A(\omega))+p_B T(\bar{G}_B(\omega))\right\}d\omega$$

It is easy to see that $\mathcal{M}\triangleq\{(\boldsymbol{p},\boldsymbol{G})$ such that $\bar{G}_m(\omega)$ is continuous decreasing with $\bar{G}_m(1)=1$ and $\bar{G}_m(\bar{\omega})=0$ and for which $\sum_{m=1}^{M} p_m(\bar{G}_m(\omega)-\bar{G}(\omega))=0$ for all $\omega\in[1,\bar{\omega}]$ and $\sum_{m\in\{A,B\}} p_m=1\}$.

□

Proof of Proposition 4 We introduce

$$L(\boldsymbol{G},\boldsymbol{p})=\sum_{m\in\{A,B\}} p_m\int_1^{\bar{\omega}} T(\bar{G}_m(\omega))d\omega \text{ and } v_m(\boldsymbol{G})=1+\int_1^{\bar{\omega}} \bar{G}_m(\omega)d\omega \text{ for } (\boldsymbol{p},\boldsymbol{G})\in\mathcal{M}.$$

and can write the objective function as

$$(\lambda-1)\times(\sum_{m\in\{A,B\}} p_m(\{v_m(\boldsymbol{G})-v^{-1}\}^+ + L(\boldsymbol{G},\boldsymbol{p}))-v^{-1}). \tag{11}$$

WLOG, we can ignore the second term v^{-1} and factor $(\lambda-1)$. For fixed $\boldsymbol{p}$, the objective is to minimize $\sum_{m\in\{A,B\}} p_m(\{v_m(\boldsymbol{G})-v^{-1}\}^+ + L(\boldsymbol{G},\boldsymbol{p})$ for $(\boldsymbol{p},\boldsymbol{G})\in\mathcal{M}$.

Key proof idea: Instead of optimizing over $(\boldsymbol{p},\boldsymbol{G})\in\mathcal{M}$, we optimize over a class of functions, $\bar{G}_m(\omega,\boldsymbol{\omega},\boldsymbol{p})$, parameterized by $\boldsymbol{\omega}=(\omega_1,\omega_2,...,\omega_{K-1})\in[1,\bar{\omega}]^{K-1}$ with $1=\omega_0\le\omega_1\le...\le\omega_{K-2}\le\omega_{K-1}\le\omega_K=\bar{\omega}$. Intuitively, we introduce a sequence $ABABABA$... with outcome A in the intervals $\cup_{k=1}^{\lceil K/2\rceil}[\omega_{2(k-1)},\omega_{2k-1}]$ and B in the other intervals. For a large enough K, the intervals can be adjusted to approximate any continuous $\bar{G}_m(\omega)$. Instead of optimizing over continuous function, we optimize over $\boldsymbol{\omega}$. We take two consecutive segments with outcome B of any $\boldsymbol{\omega}$ for $K\ge 5$ and fix the probability mass in each of these B-segments, as well as the posterior mean. Then, we show that, under some mild conditions of the prior, the objective will be improved by eliminating one of the adjacent segments with outcome A. Thus, it suffices to consider $\boldsymbol{\omega}$ with strictly less segments. For $K = 4$, an additional condition on the prior is required as fixing the probability masses of each B segment and the posterior mean uniquely specifies the thresholds. In that case, under an additional condition on the prior, we show that again the objective can be improved when fixing the total probability mass in both B-segments, as well as the posterior mean. Therefore, for any given posterior expected value, the optimal solution must have three segments or less $K \leq 3$. Finally, we show that when also optimizing over the posterior expected value, either the posterior is equal to the prior for $K = 3$ and the posterior at $t = 0$ is equal to the prior, or, $K = 2$ and the posterior for signal B (A) is strictly higher (lower) than the prior.

Posterior $\bar{G}_m(\omega, \boldsymbol{\omega}, \boldsymbol{p})$: Using Bayes' rule, the structure with outcome A in the intervals $\cup_{k=1}^{\lceil K/2 \rceil}[\omega_{2(k-1)}, \omega_{2k-1}]$ (and B in the other intervals) yield the following posterior tail distribution:

$$\bar{G}_m(\omega, \boldsymbol{\omega}, \boldsymbol{p}) = \frac{\sum_{k\text{: outcome } m \text{ in } (\omega_{k-1}, \omega_k)} \int_{\max(\omega, \omega_{k-1})}^{\max(\omega, \omega_k)} g(\omega) d\omega}{p_m}. \tag{12}$$

It is easy to see that $\bar{G}_m(\omega, \boldsymbol{\omega}, \boldsymbol{p})$ is continuous decreasing with $\bar{G}_m(1, \boldsymbol{\omega}, \boldsymbol{p}) = 1$ and $\bar{G}_m(0, \boldsymbol{\omega}, \boldsymbol{p}) = 0$ and for which $\sum_{m=1}^{M} p_m(\bar{G}_m(\omega, \boldsymbol{\omega}, \boldsymbol{p}) - \bar{G}(\omega)) = 0$ for all $\omega \in [1, \bar{\omega}]$, hence, $(\boldsymbol{p}, \boldsymbol{G}) \in \mathcal{M}$.

Optimization over $\boldsymbol{\omega}$: We can write the following optimization problem in $\boldsymbol{\omega}$:

$$\min_{\boldsymbol{\omega}} \sum_{m \in \{A,B\}} p_m \{v_m(\boldsymbol{\omega}, \boldsymbol{p}) - v^{-1}\}^+ + L(\boldsymbol{\omega}, \boldsymbol{p}) \text{ where}$$

$$L(\boldsymbol{\omega}, \boldsymbol{p}) = \sum_{m \in \{A,B\}} p_m \int_1^{\bar{\omega}} T(\bar{G}_m(\omega, \boldsymbol{p}, \boldsymbol{\omega})) d\omega \text{ and } v_m(\boldsymbol{\omega}, \boldsymbol{p}) = 1 + \int_1^{\bar{\omega}} \bar{G}_m(\omega, \boldsymbol{\omega}, \boldsymbol{p}) d\omega.$$

Introduce $\boldsymbol{v} = (v_A, v_B) \in \mathbb{R}_+^2$, where v_m is the posterior expectation of outcome m. Notice that not all $(\boldsymbol{v}, \boldsymbol{p}) \in \mathbb{R}_+^2 \times [0,1]^2$ can be achieved. For every p_A, we have that $p_B = 1 - p_A$ and $v_B = (v^{-1} - p_A v_A)/(1 - p_A)$ and it is easy to see that

$$\underline{v}_A(p_A) \triangleq 1 + \frac{\int_1^{\underline{w}(p_A)} \bar{G}(\omega) d\omega}{p_A} \leqslant v_A \leqslant \bar{v}_A(p_A) \triangleq 1 + \frac{\bar{w}(p_A) - 1 + \int_{\bar{w}(p_A)}^{\bar{\omega}} \bar{G}(\omega) d\omega}{p_A}$$
$$\text{where } \underline{w}(p_A) \text{ and } \bar{w}(p_A) \text{ solve } 1 - \bar{G}(\underline{w}) = \bar{G}(\bar{w}) = p_A$$

are the lowest and highest possible value for v_A (as all mass is compacted at the lowest and highest values respectively). Now, define

$$\mathcal{V}(\boldsymbol{p}) \triangleq \{(v_A, v_B) \in [1, \bar{\omega}] : \underline{v}_A(p_A) \leqslant v_A \leqslant \bar{v}_A(p_A) \text{ and } v_B = (v^{-1} - p_A v_A)/(1 - p_A)\}.$$

Note that $(v^{-1}, v^{-1}) \in \mathcal{V}(\boldsymbol{p})$. It is easy to see that with $K = 2$, it is impossible to obtain $v^{-1} = v_A(\omega_1, \boldsymbol{p}) = v_B(\omega_1, \boldsymbol{p})$. Fixing $\boldsymbol{v} \in \mathcal{V}(\boldsymbol{p})$, we have the following problem: we will consider the problem

$$\min_{\boldsymbol{\omega}} L(\boldsymbol{\omega}, \boldsymbol{p}) \text{ subject to } \boldsymbol{v} = (v_A(\boldsymbol{\omega}, \boldsymbol{p}), v_B(\boldsymbol{\omega}, \boldsymbol{p})).$$

From now on, we consider only $\boldsymbol{v} \in \mathcal{V}(\boldsymbol{p})$. For ease of notation, we drop $(\boldsymbol{v}, \boldsymbol{p})$ from the arguments. Now, let the mass in each of the intervals for outcome B be p_{Bk} with $\sum_{k=1}^{K_B} p_{Bk} = p_B$. First, consider $K \geqslant 5$. Consider two consecutive B-intervals, k and $k+1$, where $1 < k < K_B$. Then, consider the four thresholds for these two intervals, $(\omega_{2k-1}, \omega_{2k}, \omega_{2k+1}, \omega_{2k+2})$ and keep the other thresholds fixed. Now, we study the following minimization problem:

$$\min_{(\omega_{2k-1},\omega_{2k},\omega_{2k+1},\omega_{2k+2})} L(\boldsymbol{\omega}) \text{ s.t. } v_{B0} = v_B(\boldsymbol{\omega}),\ p_{Bk} = \bar{G}(\omega_{2k-1}, \boldsymbol{\omega}) - \bar{G}(\omega_{2k}, \boldsymbol{\omega})$$
$$\text{and } p_{Bk+1} = \bar{G}(\omega_{2k+1}, \boldsymbol{\omega}) - \bar{G}(\omega_{2k+2}, \boldsymbol{\omega}).$$

Given $(p_{Bk}, p_{Bk+1}, v_{B0})$, there are three conditions on four variables and hence, only one degree of freedom. WLOG, we let ω_{2k-1} be the independent variable and let $\hat{L}(\omega_{2k-1})$ a single-dimensional function of ω_{2k-1}. We show that when the prior, $g(\omega)$, is concave decreasing, when $\frac{d\hat{L}(\omega_{2k-1})}{d\omega_{2k-1}} = 0$, it follows that $\frac{d^2\hat{L}(\omega_{2k-1})}{d\omega_{2k-1}^2} < 0$. Therefore, the objective to be minimized is concave in ω_{2k-1} and thus $\hat{L}(\omega_{2k-1})$ cannot have an interior solution. That is, either $\omega_{2k-1}^* = \omega_{2k-2}$, or $\omega_{2k-1}^* = \omega_{2k}^*$ or $\omega_{2k+2}^* = \omega_{2k+3}$. This holds for any $(p_{Bk}, p_{Bk+1}, v_{B0})$.

Claim 1. *When $g(\omega)$ is concave, then $\hat{L}(\omega_{2k-1})$ cannot have a local minimum for $K \geqslant 5$.*

Proof See Lemmas 5, 6 and 10.

□

As a consequence, the optimal number of segments must by $K \leq 4$. We cannot follow the same proof structure for $K = 4$ because for $K = 4$, there are only three independent thresholds; $(\omega_1, \omega_2, \omega_3)$ and for given (p_{B1}, p_{B2}, v_{B0}), there thresholds are uniquely determined. For $K = 4$, we fix the *total* mass for signal B, p_B, as well as the posterior, v_B. Then consider the problem

$$\min_{(\omega_1,\omega_2,\omega_3)} L(\boldsymbol{\omega}) \text{ subject to } v_{B0} = v_B(\boldsymbol{\omega}),\ p_B = \bar{G}(\omega_1) - \bar{G}(\omega_2) + \bar{G}(\omega_3).$$

Given (p_B, v_{B0}), there are two conditions on three variables and hence, only one degree of freedom. WLOG, we let ω_2 be the independent variable and $\hat{L}(\omega_2)$ a single-dimensional function of ω_2. We show that when the prior, $g(\omega)$, is concave decreasing, when $\frac{d\hat{L}(\omega_2)}{d\omega_2} = 0$, it follows that $\frac{d^2\hat{L}(\omega_2)}{d\omega_2^2} < 0$. Therefore, the objective to be minimized is concave in ω_2 and thus $\hat{L}(\omega_2)$ cannot have an interior solution. This holds for any (p_B, v_B).

Claim 2. *When $g(\omega)$ is concave decreasing, then $\hat{L}(\omega_1)$ cannot have a local minimum for $K = 4$.*

Proof See Lemmas 7, 8 and 10.

□

Finally, recall that, for a given $\boldsymbol{p}$, the objective function is

$$\min_{\boldsymbol{\omega}} \sum_{m\in\{A,B\}} p_m\{v_m(\boldsymbol{\omega})-v^{-1}\}^+ + L(\boldsymbol{\omega}).$$

Thus, fixing the posterior, we have optimally $K \leq 3$ segments.

Claim 3. *When $g(\omega)$ is concave, optimizing over the posterior either leads to either the posterior is equal to the prior and $K=3$, or $K=2$ and the posterior is greater than the prior for $K \leqslant 3$.*

Proof See Lemmas 9 and 10.

□

Proof of Proposition 5 This proof has been argued in the main body.

□

Proof of Proposition 6 Similar to the proof of Proposition 2, we can first write out the expression for risk conscious disutility. Denote W_1, W_2 and W_3 as the expected risk conscious disutility of customers with structure AAB, ABA, and ABB, respectively.

When the structure is AAB, we can write out the explicit expression for the cumulative standard deviation. Note that only customers with starting times 1 and 2 will have uncertainty:

$$\begin{aligned}
W_1 &= (1-g_3)\sqrt{\frac{g_1}{1-g_3}(1-\frac{g_1+2(1-g_1-g_3)}{1-g_3})^2+\frac{1-g_1-g_3}{1-g_3}(2-\frac{g_1+2(1-g_1-g_3)}{1-g_3})^2+0}\\
&= (1-g_3)\sqrt{\frac{g_1}{1-g_3}(\frac{1-g_1-g_3}{1-g_3})^2+\frac{1-g_1-g_3}{1-g_3}(\frac{g_1}{1-g_3})^2}\\
&= \sqrt{g_1(1-g_1-g_3)}
\end{aligned}$$

Similarly for structure ABA:

$$\begin{aligned}
W_2 &= (g_1+g_3)\sqrt{\frac{g_1}{g_1+g_3}(1-\frac{g_1+3g_3}{g_1+g_3})^2+\frac{g_3}{g_1+g_3}(3-\frac{g_1+3g_3}{g_1+g_3})^2+0}\\
&= (g_1+g_3)\sqrt{\frac{g_1}{g_1+g_3}(\frac{-2g_3}{g_1+g_3})^2+\frac{g_3}{g_1+g_3}(\frac{2g_1}{g_1+g_3})^2}\\
&= 2\sqrt{g_1g_3}
\end{aligned}$$

For structure ABB, the customers with starting times 2 and 3 will generate outcomes at $t=1$ that fully reveal their start times, thus no uncertainty is further accumulated at $t=1$:

$$\begin{aligned}
W_3 &= (1-g_1)\sqrt{\frac{1-g_1-g_3}{1-g_1}(2-\frac{2(1-g_1-g_3)+3g_3}{1-g_1})^2+\frac{g_3}{1-g_1}(3-\frac{2(1-g_1-g_3)+3g_3}{1-g_1})^2+0}\\
&= (1-g_1)\sqrt{\frac{1-g_1-g_3}{1-g_1}(\frac{g_3}{1-g_1})^2+\frac{g_3}{1-g_1}(\frac{1-g_1-g_3}{1-g_1})^2}\\
&= \sqrt{g_3(1-g_1-g_3)}
\end{aligned}$$

Thus for W_1, W_2, and W_3,

$$\begin{aligned}
&W_1 \geq W_2 \\
&\qquad \Longleftrightarrow \sqrt{g_1(1-g_1-g_3)} \geq 2\sqrt{g_1 g_3} \\
&\qquad \Longleftrightarrow 1-g_1-g_3 \geq 4g_3 \\
&\qquad \Longleftrightarrow g_1+5g_3 \leq 1
\end{aligned}$$

$$\begin{aligned}
&W_2 \geq W_3 \\
&\qquad \Longleftrightarrow 2\sqrt{g_1 g_3} \geq \sqrt{g_3(1-g_1-g_3)} \\
&\qquad \Longleftrightarrow 4g_1 \geq 1-g_1-g_3 \\
&\qquad \Longleftrightarrow 5g_1+g_3 \leq 1
\end{aligned}$$

$$\begin{aligned}
&W_1 \geq W_3 \\
&\qquad \Longleftrightarrow \sqrt{g_1(1-g_1-g_3)} \geq \sqrt{g_3(1-g_1-g_3)} \\
&\qquad \Longleftrightarrow g_1 \geq g_3
\end{aligned}$$

These inequalities form the boundaries of the optimal outcomes.

□

Proof of Lemma 1 We first show that if every possible expectation path is monotonic, a loss-averse customer's expected utility is equal to the upper bound $\bar{U}^L = \mathbb{E}_\Omega\{u(v^{-1}-\Omega)\}$. Thus, assume that every possible expectation path is monotonic. If $\omega \leq v^{-1}$, we have

$$\begin{aligned}
U^L(\tilde{\boldsymbol{g}}^\omega) &= \sum_{t=0}^{\omega} \left\{[\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^{t-1}) - \tilde{v}^t(\tilde{\boldsymbol{g}}^t)]^+ - \lambda[\tilde{v}^t(\tilde{\boldsymbol{g}}^t) - \tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^{t-1})]^+\right\} \\
&= \sum_{t=0}^{\omega}\{\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^{t-1}) - \tilde{v}^t(\tilde{\boldsymbol{g}}^t)\} \\
&= v^{-1} - \tilde{v}^1(\tilde{\boldsymbol{g}}^1) + \sum_{t=1}^{\omega-1}\{\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^{t-1}) - \tilde{v}^t(\tilde{\boldsymbol{g}}^t)\} + \tilde{v}^{\omega-1}(\tilde{\boldsymbol{g}}^{\omega-1}) - \omega \\
&= v^{-1} - \omega
\end{aligned}$$

Similarly, if $\omega > v^{-1}$, we can show that $U^L(\tilde{\boldsymbol{g}}^\omega) = \lambda(v^{-1}-\omega)$. Therefore, we have $\mathbb{E}_\Omega U^L(\tilde{\boldsymbol{g}}^\Omega) = \mathbb{E}_\Omega\{u(v^{-1}-\Omega)\} = \bar{U}^L$.

Now we show that if a loss-averse customer's expected utility is equal to the upper bound $\bar{U}^L$, then every possible expectation path is monotonic. We prove this statement by contradiction. Thus, assume that there exists at least one expectation path that is not monotonic. If $\omega \leq v^{-1}$,

$$\begin{aligned}
U^L(\tilde{\boldsymbol{g}}^\omega) &= \sum_{t=0}^{\omega} \Big\{ [\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^t(\tilde{\boldsymbol{g}}^\omega)]^+ - \lambda[\tilde{v}^t(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega)]^+ \Big\} \\
&< \sum_{t=0}^{\omega} \Big\{ [\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^t(\tilde{\boldsymbol{g}}^\omega)]^+ - [\tilde{v}^t(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega)]^+ \Big\} \\
&= \sum_{t=0}^{\omega} \{ \tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^t(\tilde{\boldsymbol{g}}^\omega) \} \\
&= v^{-1} - \omega \\
&= u(v^{-1} - \omega)
\end{aligned}$$

where the inequality follows because $\lambda > 1$. Now, if $\omega > v^{-1}$,

$$\begin{aligned}
U^L(\tilde{\boldsymbol{g}}^\omega) &= \sum_{t=0}^{\omega} \Big\{ [\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^t(\tilde{\boldsymbol{g}}^\omega)]^+ - \lambda[\tilde{v}^t(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega)]^+ \Big\} \\
&< \sum_{t=0}^{\omega} \Big\{ \lambda[\tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^t(\tilde{\boldsymbol{g}}^\omega)]^+ - \lambda[\tilde{v}^t(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega)]^+ \Big\} \\
&= \lambda \sum_{t=0}^{\omega} \{ \tilde{v}^{t-1}(\tilde{\boldsymbol{g}}^\omega) - \tilde{v}^t(\tilde{\boldsymbol{g}}^\omega) \} \\
&= \lambda(v^{-1} - \omega) \\
&= u(v^{-1} - \omega)
\end{aligned}$$

Because this is true for at least one expectation path, $\mathbb{E}_\Omega U^L(\tilde{\boldsymbol{g}}^\Omega) < \mathbb{E}_\Omega\{u(v^{-1} - \Omega)\} = \bar{U}^L$. Therefore, monotonicity of all expectation paths is a necessary and sufficient condition for achieving the upper bound.

□

Proof of Proposition 7 Given Lemma 1, any information structure $\boldsymbol{f}$ that produces monotonic expectation paths for the expected wait times is an optimal information structure. The following definition will be useful:

Definition 2. An information structure f_ω^t is **mean-preserving** if for all ω, $v^t(\tilde{\boldsymbol{g}}^\omega) = v^{-1}$.

In the 'candelabra' structure, the forecasts reveal the most extreme waiting-times first, while the forecasts for the middle waiting times are mean-preserving. That is, in the first period waiting times $\omega = 1$ and $\bar{\omega}$ are revealed, in the second period $\omega = 2$ and $\bar{\omega} - 1$ are revealed, etc. The expected wait-times for the remaining customers in-between those extremes remain v^{-1}. We now show that given a symmetric prior distribution, there are always structures that produce the candelabra pattern, and that information structure is optimal.

We first show that the following information structure produces the candelabra. For $t = 0, 1, \ldots, \bar{\omega} - 1$, let

$$f_\omega^t = \begin{cases} 1 \text{ for } \omega = t+1, \bar{\omega} - t \leq \omega \leq \bar{\omega} \\ 0 \text{ otherwise} \end{cases}$$

We will see that generating outcome A for $\bar{\omega} - t + 1 \leq \omega \leq \bar{\omega}$ is arbitrary; generating B for any of these customers will have the same affect. We now proceed by induction. In period $t = -1$ we have the symmetric prior distribution $\boldsymbol{g}^{-1}$. Now assume that for some $t = 0 ... \bar{\omega} - 1$ we have the following posterior distributions:

For $t + 2 \leq \omega \leq \bar{\omega} - t - 1$,

$$\tilde{g}_T^t = \begin{cases} \frac{g_T^{-1}}{\sum_{\tau=t+2}^{\bar{\omega}-t-1} g_\tau^{-1}} \text{ for } t+2 \leq T \leq \bar{\omega} - t - 1 \\ 0 \text{ otherwise} \end{cases}$$

For $\omega = t + 1$ and $\omega = \bar{\omega} - t$,

$$\tilde{g}_T^t = \begin{cases} \frac{1}{2} \text{ for } T = t+1 \\ \frac{1}{2} \text{ for } T = \bar{\omega} - t \\ 0 \text{ otherwise} \end{cases}$$

Finally, given $t > 0$ and for $\omega \leq t$ and $\omega \geq \bar{\omega} - t + 1$, $\tilde{g}_T^t = 1$ for $T = \omega$, 0 otherwise. That is, the t service times on both the low and high sides of the distributions have been revealed.

Note that for $t + 1 \leq \omega \leq \bar{\omega} - t$, $v^t = v^{-1} = (1/2)(\bar{\omega} + 1)$. We now show that the information structure described above preserves this posterior distribution when we move from t to $t + 1$. At time period $t + 1$, customers $t + 3 \leq \omega \leq \bar{\omega} - t - 2$, generate outcome B, and for these customers,

$$\tilde{g}_T^{t+1} = \begin{cases} \frac{g_T^{-1}}{\sum_{\tau=t+3}^{\bar{\omega}-t-2} g_\tau^{-1}} \text{ for } t+3 \leq T \leq \bar{\omega} - t - 2 \\ 0 \text{ otherwise} \end{cases}$$

Customers with $\omega = t + 2$ and $\bar{\omega} - t - 1 \leq \omega \leq \bar{\omega}$, generate outcome A. For $\bar{\omega} - t \leq \omega \leq \bar{\omega}$, the service times have been revealed and the observed outcome has no effect. Now consider the customer with $\omega = t + 2$. From the induction assumption, for this customer $\tilde{g}_T^t = 0$ for $T > \bar{\omega} - t - 1$. Also, because the distribution is symmetric, $\tilde{g}_{t+2}^t = \tilde{g}_{\bar{\omega}-t-1}^t$. Therefore,

$$\tilde{g}_{t+2}^{t+1} = \frac{\tilde{g}_{t+2}^t}{\tilde{g}_{t+2}^t + \tilde{g}_{\bar{\omega}-t-1}^t} = \frac{1}{2}$$

We have the same result for $\tilde{g}_{\bar{\omega}-t-1}^{t+1}$, given a customer with $\omega = t + 2$, as well as the same posterior distribution for customers with $\omega = \bar{\omega} - t - 1$.

Finally, we consider the two customers at the extremes. A customer with $\omega = t + 1$ has gone into service at time $t + 1$ so that $\tilde{g}_{t+1}^{t+1} = 1$. For the customer with $\omega = \bar{\omega} - t$,

$$\tilde{g}_{\bar{\omega}-t}^{t+1} = \frac{\tilde{g}_{\bar{\omega}-t}^t}{1 - \tilde{g}_{t+1}^t} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

Therefore, for a customer with $\omega = \bar{\omega} - t$, for any outcome s, $\tilde{g}^{t+1}_{\bar{\omega}-t} = g^{t+1}_{\bar{\omega}-t}(s) = 1$. This shows that the posterior distributions are preserved. Note again that for $t+2 \leq \omega \leq \bar{\omega} - t - 1$, $v^{t+1} = v^{-1}$, so the information structure is mean-preserving. This implies that for any ω, the sequence of expected wait times consists of one or more values v^{-1}, and then ω itself. Therefore the sequence of expected wait times is monotonic, and by lemma 1, the information structure is optimal.

□

2. Auxiliary Results

Before proving Lemmas 5, 6, 7, 8, 9 and 10, we first establish auxiliary Lemmas for the closed form of the objective function, .

LEMMA 3. *For any given threshold structure $\boldsymbol{\omega}$ for $K \geq 5$ segments, we have:*

$$\begin{aligned} L(\boldsymbol{\omega}) = & p_A\{\int_1^{\omega_1} T(\frac{\bar{G}(\omega) - \bar{G}(\omega_1) + \bar{G}(\omega_2) - \bar{G}(\omega_3) + \bar{G}(\omega_4) + H}{p_A})d\omega \\ & + \int_{\omega_1}^{\omega_2} T(\frac{\bar{G}(\omega_2) - \bar{G}(\omega_3) + \bar{G}(\omega_4) + H}{p_A})d\omega \\ & + \int_{\omega_2}^{\omega_3} T(\frac{\bar{G}(\omega) - \bar{G}(\omega_3) + \bar{G}(\omega_4) + H}{p_A})d\omega + \int_{\omega_3}^{\omega_4} T(\frac{\bar{G}(\omega_4) + H}{p_A})d\omega + \underbrace{\cdots}_{independent\ of\ (\omega_1,\omega_2,\omega_3,\omega_4)} \} \\ & + p_B\{\int_1^{\omega_1} T(\underbrace{\frac{\bar{G}(\omega_1) - \bar{G}(\omega_2) + \bar{G}(\omega_3) - \bar{G}(\omega_4) - H}{p_B}}_{=1})d\omega \\ & + \int_{\omega_1}^{\omega_2} T(\frac{\bar{G}(\omega) - \bar{G}(\omega_2) + \bar{G}(\omega_3) - \bar{G}(\omega_4) - H}{p_B})d\omega \\ & + \int_{\omega_2}^{\omega_3} T(\frac{\bar{G}(\omega_3) - \bar{G}(\omega_4) - H}{p_B})d\omega + \int_{\omega_3}^{\omega_4} T(\frac{\bar{G}(\omega) - \bar{G}(\omega_4) - H}{p_B})d\omega + \underbrace{\cdots}_{independent\ of\ (\omega_1,\omega_2,\omega_3,\omega_4)} \} \end{aligned}$$

WLOG, we can focus on $K = 6$ (by considering the first six segments of any structure with $K > 6$) and transform variables $\boldsymbol{\omega}$ into $\boldsymbol{\gamma}$ as follows:

$$\bar{G}(\omega) = \gamma,\ h(\gamma) = \frac{1}{g(\bar{G}^{-1}(\gamma))} \text{ and}$$
$$1 = \gamma_6 > \bar{G}(\omega_1) = \gamma_5 > \bar{G}(\omega_2) = \gamma_4 > \bar{G}(\omega_3) = \gamma_3 > \bar{G}(\omega_4) = \gamma_2 > \bar{G}(\omega_5) = \gamma_1 > 0 = \gamma_0.$$

This transformation makes the derivations below cleaner. Now, with a slight abuse of notation, we rewrite $L(\boldsymbol{\omega})$ as $L(\boldsymbol{\gamma})$:

LEMMA 4. *For any given threshold structure $\boldsymbol{\gamma}$ for $K = 6$ segments, the objective function is given by:*

$$L(\boldsymbol{\gamma}) = p_A\{\int_{\gamma_5}^{1} T(\frac{\gamma - \gamma_5 + \gamma_4 - \gamma_3 + \gamma_2 - \gamma_1}{p_A})h(\gamma)d\gamma + \int_{\gamma_4}^{\gamma_5} T(\frac{\gamma_4 - \gamma_3 + \gamma_2 - \gamma_1}{p_A})h(\gamma)d\gamma$$

$$+\int_{\gamma_3}^{\gamma_4} T(\frac{\gamma-\gamma_3+\gamma_2-\gamma_1}{p_A})h(\gamma)d\gamma+\int_{\gamma_2}^{\gamma_3} T(\frac{\gamma_2-\gamma_1}{p_A})h(\gamma)d\gamma+\int_{\gamma_1}^{\gamma_2} T(\frac{\gamma-\gamma_1}{p_A})h(\gamma)d\gamma\}$$
$$+p_B\{\int_{\gamma_5}^{1} T\underbrace{(\frac{\gamma_5-\gamma_4+\gamma_3-\gamma_2+\gamma_1}{p_B})}_{=1}h(\gamma)d\gamma$$
$$+\int_{\gamma_4}^{\gamma_5} T(\frac{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1}{p_B})h(\gamma)d\gamma+\int_{\gamma_3}^{\gamma_4} T(\frac{\gamma_3-\gamma_2+\gamma_1}{p_B})h(\gamma)d\gamma+\int_{\gamma_2}^{\gamma_3} T(\frac{\gamma-\gamma_2+\gamma_1}{p_B})h(\gamma)d\gamma$$
$$+\int_{\gamma_2}^{\gamma_3} T(\frac{\gamma_1}{p_B})h(\gamma)d\gamma+\int_{0}^{\gamma_1} T(\frac{\gamma}{p_B})h(\gamma)d\gamma\}$$

and the posterior expected wait time for structure B, $v_B(\boldsymbol{\gamma})$, is

$$v_B(\boldsymbol{\gamma})=1+\int_{\gamma_5}^{1}\frac{\gamma_5-\gamma_4+\gamma_3-\gamma_2+\gamma_1}{p_B}h(\gamma)d\gamma+\int_{\gamma_4}^{\gamma_5}\frac{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1}{p_B}h(\gamma)d\gamma$$
$$+\int_{\gamma_3}^{\gamma_4}\frac{\gamma_3-\gamma_2+\gamma_1}{p_B}h(\gamma)d\gamma+\int_{\gamma_2}^{\gamma_3}\frac{\gamma-\gamma_2+\gamma_1}{p_B}h(\gamma)d\gamma+\int_{\gamma_1}^{\gamma_2}\frac{\gamma_1}{p_B}h(\gamma)d\gamma+\int_{0}^{\gamma_1}\frac{\gamma}{p_B}h(\gamma)d\gamma$$

and $v_A(\boldsymbol{\gamma})$ can be obtained from $p_A v_A(\boldsymbol{\gamma})+(1-p_A)v_B(\boldsymbol{\gamma})=v^{-1}$.

LEMMA 5. *For $K=6$, we change γ_5, keeping p_{B1}, p_{B2} and the posterior constant; $p_{B1}=\gamma_5-\gamma_4$, $p_{B2}=\gamma_3-\gamma_2$ and $v_{B0}=v_B(\boldsymbol{\gamma})$, we have:*

$$\frac{d\hat{L}(\gamma_5)}{d\gamma_5}=Z(\gamma_2,\gamma_3,\gamma_4,\gamma_5)\times\int_{\gamma_4}^{\gamma_5}h(\gamma)d\gamma \text{ where}$$
$$Z(\gamma_2,\gamma_3,\gamma_4,\gamma_5)=-\frac{\int_{\gamma_4}^{\gamma_5}\ln(\frac{\gamma_4-\gamma_3+\gamma_2-\gamma_1}{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_4}^{\gamma_5}h(\gamma)d\gamma}+\frac{\int_{\gamma_2}^{\gamma_3}\ln(\frac{\gamma_2-\gamma_1}{\gamma-\gamma_2+\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}$$

LEMMA 6. *For $K=6$, we change γ_5, keeping p_{B1}, p_{B2} and the posterior constant; $p_{B1}=\gamma_5-\gamma_4$, $p_{B2}=\gamma_3-\gamma_2$ and $v_{B0}=v_B(\boldsymbol{\gamma})$, we have that when $\frac{d\hat{L}(\gamma_5)}{d\gamma_5}=0$ then*

$$\text{when } h(\gamma) \text{ is concave } \Rightarrow \frac{d^2\hat{L}(\gamma_5)}{d\gamma_5^2}<0$$

LEMMA 7. *For $K=4$, we change γ_3, keeping p_B and the posterior constant; $p_B=\gamma_3-\gamma_2+\gamma_1$ and $v_{B0}=v_B(\boldsymbol{\gamma})$, we have that:*

$$\frac{d\hat{L}(\gamma_2)}{d\gamma_2}=Z(\gamma_1,\gamma_2,\gamma_3)\times\frac{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3}h(\gamma)d\gamma}$$
$$\text{where } Z(\gamma_1,\gamma_2,\gamma_3)=\frac{\int_{\gamma_1}^{\gamma_2}\ln(\frac{\gamma-\gamma_1}{\gamma_2-\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}+\frac{\int_{\gamma_2}^{\gamma_3}\ln(\frac{\gamma-\gamma_2+\gamma_1}{\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}$$

LEMMA 8. *For $K=4$, we change γ_3, keeping p_B and the posterior constant; $p_B=\gamma_3-\gamma_2+\gamma_1$ and $v_{B0}=v_B(\boldsymbol{\gamma})$, we have that when $\frac{d\hat{L}(\gamma_2)}{d\gamma_2}=0$ then*

$$\text{when } h(\gamma) \text{ is concave decreasing } \Rightarrow \frac{d^2\hat{L}(\gamma_2)}{d\gamma_2^2}<0$$

LEMMA 9. *For $K=3$, we change γ_2, keeping p_B, $p_B=\gamma_2-\gamma_1$ and $v_{B0}=v_B(\boldsymbol{\gamma})$, we have the following necessary condition for a local optimum:* $\frac{d\hat{L}(\gamma_2)}{d\gamma_2}=0$ *with*

$$\frac{d\hat{L}(\gamma_2)}{d\gamma_2}=\int_{\gamma_1}^{\gamma_2}\left\{1-\ln(\frac{\gamma_1}{\gamma-\gamma_1})+\ln(\frac{p_A}{1-p_A})\right\}h(\gamma)d\gamma$$

and when $\frac{d\hat{L}(\gamma_2)}{d\gamma_2}=0$ *then*

$$\textit{when } h(\gamma) \textit{ is concave } \Rightarrow \frac{d^2\hat{L}(\gamma_2)}{d\gamma_2^2}<0$$

LEMMA 10. *When $g(\omega)$ is concave decreasing, $h(\gamma)$ is concave decreasing too.*

3. Proofs of Auxiliary results

Proof of Lemma 3 : With

$$\bar{G}_A(\omega,\boldsymbol{\omega})=\frac{\int_{\omega\in\cup_{k=1}^{K_A}[\omega_{2(k-1)},\omega_{2k-1}]}g(\omega)d\omega}{p_A}=\frac{\sum_{k=1}^{K_A}\int_{\max(\omega,\omega_{2(k-1)})}^{\max(\omega,\omega_{2k-1})}g(\omega)d\omega}{p_A}$$
$$=\frac{\sum_{k=1}^{K_A}(\bar{G}(\max(\omega,\omega_{2(k-1)}))-\bar{G}(\max(\omega,\omega_{2k-1})))}{p_A}$$

the objective function is:

$$p_A\int_1^{\bar{\omega}}T(\frac{\sum_{k=1}^{K_A}(\bar{G}(\max(\omega,\omega_{2(k-1)}))-\bar{G}(\max(\omega,\omega_{2k-1})))}{p_A})d\omega$$
$$+(1-p_A)\int_1^{\bar{\omega}}T(\frac{\sum_{k=1}^{K_B}(\bar{G}(\max(\omega,\omega_{2k-1}))-\bar{G}(\max(\omega,\omega_{2k})))}{1-p_A})d\omega$$

or the integrands can be written as:

$$\begin{cases}\frac{\bar{G}(\omega)-\bar{G}(\omega_1)+\bar{G}(\omega_2)-\bar{G}(\omega_3)+\bar{G}(\omega_4)-\bar{G}(\omega_5)+\ldots+\bar{G}(\omega_{K_A-1})-\bar{G}(\omega_{K_A})}{p_A} & 1\leqslant\omega\leqslant\omega_1\\ \frac{\bar{G}(\omega_2)-\bar{G}(\omega_3)+\bar{G}(\omega_4)-\bar{G}(\omega_5)+\ldots+\bar{G}(\omega_{K_A-1})-\bar{G}(\omega_{K_A})}{p_A} & \omega_1\leqslant\omega\leqslant\omega_2\\ \frac{\bar{G}(\omega)-\bar{G}(\omega_3)+\bar{G}(\omega_4)-\bar{G}(\omega_5)+\ldots+\bar{G}(\omega_{K_A-1})-\bar{G}(\omega_{K_A})}{p_A} & \omega_2\leqslant\omega\leqslant\omega_3\\ \frac{\bar{G}(\omega_4)-\bar{G}(\omega_5)+\ldots+\bar{G}(\omega_{K_A-1})-\bar{G}(\omega_{K_A})}{p_A} & \omega_3\leqslant\omega\leqslant\omega_4\\ \vdots & \vdots\end{cases}$$

and

$$\begin{cases}\frac{\bar{G}(\omega_1)-\bar{G}(\omega_2)+\bar{G}(\omega_3)-\bar{G}(\omega_4)+\bar{G}(\omega_5)-\bar{G}(\omega_6)+\ldots+\bar{G}(\omega_{K_B-1})-\bar{G}(\omega_{K_B})}{1-p_A} & 1\leqslant\omega\leqslant\omega_1\\ \frac{\bar{G}(\omega)-\bar{G}(\omega_2)+\bar{G}(\omega_3)-\bar{G}(\omega_4)+\bar{G}(\omega_5)-\bar{G}(\omega_6)+\ldots+\bar{G}(\omega_{K_B-1})-\bar{G}(\omega_{K_B})}{1-p_A} & \omega_1\leqslant\omega\leqslant\omega_2\\ \frac{\bar{G}(\omega_3)-\bar{G}(\omega_4)+\bar{G}(\omega_5)-\bar{G}(\omega_6)+\ldots+\bar{G}(\omega_{K_B-1})-\bar{G}(\omega_{K_B})}{1-p_A} & \omega_2\leqslant\omega\leqslant\omega_3\\ \frac{\bar{G}(\omega)-\bar{G}(\omega_4)+\bar{G}(\omega_5)-\bar{G}(\omega_6)+\ldots+\bar{G}(\omega_{K_B-1})-\bar{G}(\omega_{K_B})}{1-p_A} & \omega_3\leqslant\omega\leqslant\omega_4\\ \vdots & \vdots\end{cases}$$

or with

$$H(\omega_5,\ldots)=-\bar{G}(\omega_5)+\sum_{k=4}^{K_A}(\bar{G}(\omega_{2(k-1)})-\bar{G}(\omega_{2k-1}))$$

we can write

$$\begin{cases} \frac{\bar{G}(\omega)-\bar{G}(\omega_1)+\bar{G}(\omega_2)-\bar{G}(\omega_3)+\bar{G}(\omega_4)+H(\omega_5,\ldots)}{p_A} & 1 \leqslant \omega \leqslant \omega_1 \\ \frac{\bar{G}(\omega_2)-\bar{G}(\omega_3)+\bar{G}(\omega_4)+H(\omega_5,\ldots)}{p_A} & \omega_1 \leqslant \omega \leqslant \omega_2 \\ \frac{\bar{G}(\omega)-\bar{G}(\omega_3)+\bar{G}(\omega_4)+H(\omega_5,\ldots)}{p_A} & \omega_2 \leqslant \omega \leqslant \omega_3 \\ \frac{H(\omega_5,\ldots)}{p_A} & \omega_3 \leqslant \omega \leqslant \omega_4 \\ \vdots & \vdots \end{cases}$$

and

$$\begin{cases} \frac{\bar{G}(\omega_1)-\bar{G}(\omega_2)+\bar{G}(\omega_3)-\bar{G}(\omega_4)+\bar{G}(\omega_4)-H(\omega_5,\ldots)}{1-p_A} & 1 \leqslant \omega \leqslant \omega_1 \\ \frac{\bar{G}(\omega)-\bar{G}(\omega_2)+\bar{G}(\omega_3)-\bar{G}(\omega_4)-H(\omega_5,\ldots)}{1-p_A} & \omega_1 \leqslant \omega \leqslant \omega_2 \\ \frac{\bar{G}(\omega_3)-\bar{G}(\omega_4)-H(\omega_5,\ldots)}{1-p_A} & \omega_2 \leqslant \omega \leqslant \omega_3 \\ \frac{\bar{G}(\omega)-H(\omega_5,\ldots)}{1-p_A} & \omega_3 \leqslant \omega \leqslant \omega_4 \\ \vdots & \vdots \end{cases}$$

and after dropping the arguments $(\omega_5,...,\omega_{K-1})$.

$$\begin{aligned} L(\boldsymbol{\omega}) =& p_A\{\int_1^{\omega_1} T(\frac{\bar{G}(\omega)-\bar{G}(\omega_1)+\bar{G}(\omega_2)-\bar{G}(\omega_3)+\bar{G}(\omega_4)+H}{p_A})d\omega \\ &+\int_{\omega_1}^{\omega_2} T(\frac{\bar{G}(\omega_2)-\bar{G}(\omega_3)+\bar{G}(\omega_4)+H}{p_A})d\omega \\ &+\int_{\omega_2}^{\omega_3} T(\frac{\bar{G}(\omega)-\bar{G}(\omega_3)+\bar{G}(\omega_4)+H}{p_A})d\omega + \int_{\omega_3}^{\omega_4} T(\frac{\bar{G}(\omega_4)+H}{p_A})d\omega + \underbrace{\ldots}_{\text{independent of }(\omega_1,\omega_2,\omega_3,\omega_4)}\} \\ &+(1-p_A)\{\int_1^{\omega_1} T(\underbrace{\frac{\bar{G}(\omega_1)-\bar{G}(\omega_2)+\bar{G}(\omega_3)-\bar{G}(\omega_4)-H}{1-p_A}}_{=1})d\omega \\ &+\int_{\omega_1}^{\omega_2} T(\frac{\bar{G}(\omega)-\bar{G}(\omega_2)+\bar{G}(\omega_3)-\bar{G}(\omega_4)-H}{1-p_A})d\omega \\ &+\int_{\omega_2}^{\omega_3} T(\frac{\bar{G}(\omega_3)-\bar{G}(\omega_4)-H}{1-p_A})d\omega + \int_{\omega_3}^{\omega_4} T(\frac{\bar{G}(\omega)-\bar{G}(\omega_4)-H}{1-p_A})d\omega + \underbrace{\ldots}_{\text{independent of }(\omega_1,\omega_2,\omega_3,\omega_4)}\} \end{aligned}$$

(Note that $p_A = 1-\bar{G}(\omega_1)+\bar{G}(\omega_2)-\bar{G}(\omega_3)+\bar{G}(\omega_4)+H$.)

□

Proof of Lemma 4 : Follows from straightforward substitution of $\bar{G}(\omega)=\gamma$ in the objective function in Lemma 11.

□

Proof of Lemma 5 : We have $p_{B1}=\gamma_5-\gamma_4$, $p_{B2}=\gamma_3-\gamma_2$ and $v_B(\boldsymbol{\gamma})=v_{B0}$ and from deriving these conditions wrt γ_5, we obtain

$$\frac{d\gamma_4}{d\gamma_5}=1 \text{ and } \frac{d\gamma_2}{d\gamma_5}=\frac{d\gamma_3}{d\gamma_5}=-\frac{\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}$$

The first order derivative wrt of $\hat{L}(\gamma_5)$ wrt γ_5 is:

$$\frac{d\hat{L}}{d\gamma_5} = \frac{\partial L}{\partial \gamma_5} + \frac{\partial L}{\partial \gamma_4}\frac{d\gamma_4}{d\gamma_5} + \frac{\partial L}{\partial \gamma_3}\frac{d\gamma_3}{d\gamma_5} + \frac{\partial L}{\partial \gamma_2}\frac{d\gamma_2}{d\gamma_5}$$

or, after some straightforward algebra:

$$\frac{\partial L}{\partial \gamma_5} + \frac{\partial L}{\partial \gamma_4}\frac{d\gamma_4}{d\gamma_5} = \int_{\gamma_4}^{\gamma_5} T'(\frac{\gamma_4-\gamma_3+\gamma_2-\gamma_1}{p_A})h(\gamma)d\gamma - \int_{\gamma_4}^{\gamma_5} T'(\frac{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1}{1-p_A})h(\gamma)d\gamma$$

$$\frac{\partial L}{\partial \gamma_3}\frac{d\gamma_3}{d\gamma_5} + \frac{\partial L}{\partial \gamma_2}\frac{d\gamma_2}{d\gamma_5} = -\left\{\int_{\gamma_2}^{\gamma_3} T'(\frac{\gamma_2-\gamma_1}{p_A})h(\gamma)d\gamma - \int_{\gamma_2}^{\gamma_3} T'(\frac{\gamma-\gamma_2+\gamma_1}{1-p_A})h(\gamma)d\gamma\right\}\frac{\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}$$

and thus

$$\begin{aligned}\frac{d\hat{L}}{d\gamma_5} = &\int_{\gamma_4}^{\gamma_5} T'(\frac{\gamma_4-\gamma_3+\gamma_2-\gamma_1}{p_A})h(\gamma)d\gamma - \int_{\gamma_4}^{\gamma_5} T'(\frac{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1}{1-p_A})h(\gamma)d\gamma \\ &-\left\{\int_{\gamma_2}^{\gamma_3} T'(\frac{\gamma_2-\gamma_1}{p_A})h(\gamma)d\gamma - \int_{\gamma_2}^{\gamma_3} T'(\frac{\gamma-\gamma_2+\gamma_1}{1-p_A})h(\gamma)d\gamma\right\}\frac{\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}\end{aligned}$$

and with $T'(G) = -\ln(G) - 1$, we have, we can write after some straightforward algebra:

$$\begin{aligned}\frac{d\hat{L}}{d\gamma_5} &= -\left\{\int_{\gamma_4}^{\gamma_5} \ln(\frac{\gamma_4-\gamma_3+\gamma_2-\gamma_1}{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1})h(\gamma)d\gamma - \int_{\gamma_4}^{\gamma_5} \ln(\frac{p_A}{1-p_A})h(\gamma)d\gamma\right\} \\ &\quad + \left\{\int_{\gamma_2}^{\gamma_3} \ln(\frac{\gamma_2-\gamma_1}{\gamma-\gamma_2+\gamma_1})h(\gamma)d\gamma - \int_{\gamma_2}^{\gamma_3} \ln(\frac{p_A}{1-p_A})h(\gamma)d\gamma\right\}\frac{\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} \\ &= -\int_{\gamma_4}^{\gamma_5} \ln(\frac{\gamma_4-\gamma_3+\gamma_2-\gamma_1}{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1})h(\gamma)d\gamma + \int_{\gamma_2}^{\gamma_3} \ln(\frac{\gamma_2-\gamma_1}{\gamma-\gamma_2+\gamma_1})h(\gamma)d\gamma\frac{\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} \\ &= \left\{\underbrace{-\frac{\int_{\gamma_4}^{\gamma_5} \ln(\frac{\gamma_4-\gamma_3+\gamma_2-\gamma_1}{\gamma-\gamma_4+\gamma_3-\gamma_2+\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma} + \frac{\int_{\gamma_2}^{\gamma_3} \ln(\frac{\gamma_2-\gamma_1}{\gamma-\gamma_2+\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}}_{=Z(\gamma_2,\gamma_3,\gamma_4,\gamma_5)}\right\}\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma\end{aligned}$$

□

Proof of Lemma 6 : We need to derive with respect to γ_5 again, when $Z = 0$:

$$\frac{d^2\hat{L}}{d\gamma_5^2} = \frac{dZ}{d\gamma_5} \times \int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma + \underbrace{Z}_{=0} \times \frac{d}{d\gamma_5}\int_{\gamma_4}^{\gamma_5} h(\gamma)d\gamma$$

and when $Z = 0$, the sign of $\frac{d^2\hat{L}}{d\gamma_5^2}$ is determined by the sign of

$$\frac{dZ}{d\gamma_5} = \frac{\partial Z}{\partial \gamma_5} + \frac{\partial Z}{\partial \gamma_4}\frac{d\gamma_4}{d\gamma_5} + \frac{\partial Z}{\partial \gamma_3}\frac{d\gamma_3}{d\gamma_5} + \frac{\partial Z}{\partial \gamma_2}\frac{d\gamma_2}{d\gamma_5}.$$

Now, we rewrite $Z(\gamma_2,\gamma_3,\gamma_4,\gamma_5)$ introducing

$$Y(\underline{\gamma},\bar{\gamma},\Delta) = \frac{\int_{\underline{\gamma}}^{\bar{\gamma}} \ln(\frac{\underline{\gamma}-\Delta}{\gamma-\underline{\gamma}+\Delta})h(\gamma)d\gamma}{\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma)d\gamma}.$$

With $\Delta = \gamma_3 - \gamma_2 + \gamma_1$ and $\gamma_4 = \underline{\gamma}$ and $\gamma_5 = \bar{\gamma}$ in the first term and $\Delta = \gamma_1$ and $\gamma_2 = \underline{\gamma}$ and $\gamma_3 = \bar{\gamma}$ in the second term, it is easy to see that

$$Z(\gamma_2, \gamma_3, \gamma_4, \gamma_5) = -Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1) + Y(\gamma_2, \gamma_3, \gamma_1).$$

Thus, we have the following partial derivatives:

$$\frac{\partial Z}{\partial \gamma_5} = -\frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \bar{\gamma}}, \ \frac{\partial Z}{\partial \gamma_4} = -\frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \underline{\gamma}}$$

$$\frac{\partial Z}{\partial \gamma_3} = \frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \Delta} + \frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \bar{\gamma}} \text{ and } \frac{\partial Z}{\partial \gamma_2} = -\frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \Delta} + \frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \underline{\gamma}}$$

and

$$\begin{aligned}
\frac{dZ}{d\gamma_5} &= \frac{\partial Z}{\partial \gamma_5} + \frac{\partial Z}{\partial \gamma_4} - \left\{ \frac{\partial Z}{\partial \gamma_3} + \frac{\partial Z}{\partial \gamma_2} \right\} \frac{\int_{\gamma_4}^{\gamma_5} h(\gamma) d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma) d\gamma} \\
&= -\frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \bar{\gamma}} + \frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \underline{\gamma}} \\
&\quad - \left\{ \begin{array}{l} -\frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \Delta} + \frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \bar{\gamma}} \\ +\frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \Delta} + \frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \underline{\gamma}} \end{array} \right\} \frac{\int_{\gamma_4}^{\gamma_5} h(\gamma) d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma) d\gamma} \\
&= -\left\{ \frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \bar{\gamma}} + \frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \underline{\gamma}} \right\} \\
&\quad - \left\{ \frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \bar{\gamma}} + \frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \underline{\gamma}} \right\} \frac{\int_{\gamma_4}^{\gamma_5} h(\gamma) d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma) d\gamma}
\end{aligned}$$

Notice that $\frac{dZ}{d\gamma_5}$ does not depend on $\frac{\partial Y(\underline{\gamma}, \bar{\gamma}, \Delta)}{\partial \Delta}$, only on $\frac{\partial}{\partial \underline{\gamma}} Y(\underline{\gamma}, \bar{\gamma}, \Delta)$ and $\frac{\partial}{\partial \bar{\gamma}} Y(\underline{\gamma}, \bar{\gamma}, \Delta)$. These are:

$$\begin{aligned}
\frac{\partial}{\partial \underline{\gamma}} Y(\underline{\gamma}, \bar{\gamma}, \Delta) &= h(\underline{\gamma}) \frac{\int_{\underline{\gamma}}^{\bar{\gamma}} \ln(\frac{\underline{\gamma} - \Delta}{\gamma - \underline{\gamma} + \Delta}) h(\gamma) d\gamma}{(\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma) d\gamma)^2} + \frac{-\ln(\frac{\underline{\gamma} - \Delta}{\Delta}) h(\underline{\gamma}) + \int_{\underline{\gamma}}^{\bar{\gamma}} \left\{ \frac{1}{\underline{\gamma} - \Delta} + \frac{1}{\gamma - \underline{\gamma} + \Delta} \right\} h(\gamma) d\gamma}{\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma) d\gamma} \\
\frac{\partial}{\partial \bar{\gamma}} Y(\underline{\gamma}, \bar{\gamma}, \Delta) &= -h(\bar{\gamma}) \frac{\int_{\underline{\gamma}}^{\bar{\gamma}} \ln(\frac{\underline{\gamma} - \Delta}{\gamma - \underline{\gamma} + \Delta}) h(\gamma) d\gamma}{(\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma) d\gamma)^2} + \frac{\ln(\frac{\underline{\gamma} - \Delta}{\bar{\gamma} - \underline{\gamma} + \Delta}) h(\bar{\gamma})}{\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma) d\gamma}.
\end{aligned}$$

Thus as $\int_{\gamma_4}^{\gamma_5} h(\gamma) d\gamma / \int_{\gamma_2}^{\gamma_3} h(\gamma) d\gamma > 0$, a sufficient condition for $\frac{dZ}{d\gamma_5} < 0$ is that both terms $\frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \bar{\gamma}} + \frac{\partial Y(\gamma_4, \gamma_5, \gamma_3 - \gamma_2 + \gamma_1)}{\partial \underline{\gamma}}$ and $\frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \bar{\gamma}} + \frac{\partial Y(\gamma_2, \gamma_3, \gamma_1)}{\partial \underline{\gamma}}$ are positive. After some algebraic manipulation, we obtain:

$$\begin{aligned}
&\frac{\partial}{\partial \underline{\gamma}} Y(\underline{\gamma}, \bar{\gamma}, \Delta) + \frac{\partial}{\partial \bar{\gamma}} Y(\underline{\gamma}, \bar{\gamma}, \Delta) \\
&= \frac{1}{\underline{\gamma} - \Delta} + \int_{\underline{\gamma}}^{\bar{\gamma}} \left\{ -\frac{h'(\gamma)}{h(\gamma)} \int_{\underline{\gamma}}^{\bar{\gamma}} h(\tilde{\gamma}) d\tilde{\gamma} + h(\bar{\gamma}) - h(\underline{\gamma}) \right\} \ln(\gamma - \underline{\gamma} + \Delta) h(\gamma) d\gamma \frac{1}{(\int_{\underline{\gamma}}^{\bar{\gamma}} h(\tilde{\gamma}) d\tilde{\gamma})^2}.
\end{aligned}$$

A sufficient condition for $\frac{dZ}{d\gamma_5} < 0$ is that

$$-\frac{1}{\underline{\gamma} - \Delta} + \int_{\underline{\gamma}}^{\bar{\gamma}} \left\{ \frac{h'(\gamma)}{h(\gamma)} \int_{\underline{\gamma}}^{\bar{\gamma}} h(\tilde{\gamma}) d\tilde{\gamma} - (h(\bar{\gamma}) - h(\underline{\gamma})) \right\} \ln(\gamma - \underline{\gamma} + \Delta) h(\gamma) d\gamma \frac{1}{(\int_{\underline{\gamma}}^{\bar{\gamma}} h(\tilde{\gamma}) d\tilde{\gamma})^2} < 0 \quad (13)$$

So, after some algebraic manipulation, we obtain that $\frac{dZ}{d\gamma_5} < 0$ when:

$$-(\underline{\gamma}-\Delta)(h(\bar{\gamma})-h(\underline{\gamma}))\int_{\underline{\gamma}}^{\bar{\gamma}} \ln(\gamma-(\underline{\gamma}-\Delta))h(\gamma)d\gamma$$
$$+(\underline{\gamma}-\Delta)(\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma)d\gamma)\int_{\underline{\gamma}}^{\bar{\gamma}} \ln(\gamma-(\underline{\gamma}-\Delta))h'(\gamma)d\gamma-(\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma)d\gamma)^2 \leq 0$$

When $\bar{\gamma}=\underline{\gamma}$, the expression is equal to zero. When $\bar{\gamma}>\underline{\gamma}$, we show next that the derivative of the expression above wrt $\bar{\gamma}$ is negative when $h(\gamma)$ is concave, hence, the expression itself is negative:

$$\begin{aligned}
&-(\underline{\gamma}-\Delta)h'(\bar{\gamma})\int_{\underline{\gamma}}^{\bar{\gamma}} \ln(\gamma-(\underline{\gamma}-\Delta))h(\gamma)d\gamma-(\underline{\gamma}-\Delta)(h(\bar{\gamma})-h(\underline{\gamma}))\ln(\bar{\gamma}-(\underline{\gamma}-\Delta))h(\bar{\gamma})\\
&+(\underline{\gamma}-\Delta)h(\bar{\gamma})\int_{\underline{\gamma}}^{\bar{\gamma}} \ln(\gamma-(\underline{\gamma}-\Delta))h'(\gamma)d\gamma+(\underline{\gamma}-\Delta)(\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma)d\gamma)\ln(\bar{\gamma}-(\underline{\gamma}-\Delta))h'(\bar{\gamma})\\
&-2(h(\bar{\gamma})\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma)d\gamma)\leq 0
\end{aligned}$$

or

$$(\underline{\gamma}-\Delta)\int_{\underline{\gamma}}^{\bar{\gamma}}\left\{\frac{h'(\bar{\gamma})}{h(\bar{\gamma})}-\frac{h'(\gamma)}{h(\gamma)}\right\}\ln(\frac{\bar{\gamma}-(\underline{\gamma}-\Delta)}{\gamma-(\underline{\gamma}-\Delta)})h(\bar{\gamma})h(\gamma)d\gamma-2(h(\bar{\gamma})\int_{\underline{\gamma}}^{\bar{\gamma}} h(\gamma)d\gamma)\leq 0$$

As $\ln(\frac{\bar{\gamma}-(\underline{\gamma}-\Delta)}{\gamma-(\underline{\gamma}-\Delta)})>0$ for $\underline{\gamma}\leq\gamma\leq\bar{\gamma}$, the first term is negative when,

$$\frac{h'(\bar{\gamma})}{h(\bar{\gamma})}-\frac{h'(\gamma)}{h(\gamma)}<0 \Leftrightarrow \frac{h'(\gamma)}{h(\gamma)} \text{ is decreasing } \Leftrightarrow \frac{h''(\gamma)h(\gamma)-(h'(\gamma))^2}{(h(\gamma))^2}<0 \Leftarrow h''(\gamma)<0.$$

Thus, when $h(\gamma)$ is concave, the condition is satisfied.

□

Proof of Lemma 7 : We have that objective function:

$$\begin{aligned}
L(\gamma_1,\gamma_2,\gamma_3) =& p_A\{\int_{\gamma_3}^{1} T(\frac{\gamma-\gamma_3+\gamma_2-\gamma_1}{p_A})h(\gamma)d\gamma+\int_{\gamma_2}^{\gamma_3} T(\frac{\gamma_2-\gamma_1}{p_A})d\omega+\int_{\gamma_1}^{\gamma_2} T(\frac{\gamma-\gamma_1}{p_A})h(\gamma)d\gamma\}\\
&+(1-p_A)\{\int_{\gamma_2}^{\gamma_3} T(\frac{\gamma-\gamma_2+\gamma_1}{1-p_A})h(\gamma)d\gamma+\int_{\gamma_1}^{\gamma_2} T(\frac{\gamma_1}{1-p_A})h(\gamma)d\gamma+\int_{0}^{\gamma_1} T(\frac{\gamma}{1-p_A})h(\gamma)d\gamma\}
\end{aligned}$$

and constraints $p_B=\gamma_3-\gamma_2+\gamma_1$ and $v_{B0}=v_B(\boldsymbol{\gamma})=\int_{\gamma_2}^{\gamma_3}\frac{\gamma-\gamma_2+\gamma_1}{p_B}h(\gamma)d\gamma+\int_{\gamma_1}^{\gamma_2}\frac{\gamma_1}{p_B}h(\gamma)d\gamma+\int_0^{\gamma_1}\frac{\gamma}{p_B}h(\gamma)d\gamma$, from which

$$\frac{d\gamma_3}{d\gamma_2}=\frac{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3} h(\gamma)d\gamma} \text{ and } \frac{d\gamma_1}{d\gamma_2}=\frac{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3} h(\gamma)d\gamma}$$

We derive

$$\frac{d\hat{L}}{d\gamma_2}=\frac{\partial L}{\partial\gamma_1}\frac{d\gamma_1}{d\gamma_2}+\frac{\partial L}{\partial\gamma_2}+\frac{\partial L}{\partial\gamma_3}\frac{d\gamma_3}{d\gamma_2}$$

or via straighforward algebra, we obtain:

$$\frac{d\hat{L}}{d\gamma_2}=\left\{\underbrace{\frac{\int_{\gamma_1}^{\gamma_2}\ln(\frac{\gamma-\gamma_1}{\gamma_2-\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}+\frac{\int_{\gamma_2}^{\gamma_3}\ln(\frac{\gamma-\gamma_2+\gamma_1}{\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}}_{=Z(\gamma_1,\gamma_2,\gamma_3)}\right\}\times\frac{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3}h(\gamma)d\gamma}$$

$\square$

Proof of Lemma 8 : WLOG, select γ_2 as independent variable. We need that when $\frac{d\hat{L}}{d\gamma_2}=0$, then

$$\frac{d^2\hat{L}}{d\gamma_2^2}=\frac{d}{d\gamma_2}\left\{Z\times\frac{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3}h(\gamma)d\gamma}\right\}=\frac{dZ}{d\gamma_2}\times\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma+\underbrace{Z}_{=0}\times\frac{d}{d\gamma_2}\left\{\frac{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3}h(\gamma)d\gamma}\right\}<0$$

Hence, $\frac{d^2\hat{L}}{d\gamma_2^2}<0$ when $\frac{dZ}{d\gamma_2}<0$. With $\gamma_2-\gamma_1=\gamma_3-p_B$, we have

$$Z(\gamma_1,\gamma_2,\gamma_3)=\frac{\int_{\gamma_1}^{\gamma_2}\ln(\frac{\gamma-\gamma_1}{\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}+\frac{\int_{\gamma_2}^{\gamma_3}\ln(\frac{\gamma-(\gamma_3-p_B)}{(\gamma_3-p_B)})h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}$$

Now, we rewrite $Z(\gamma_1,\gamma_2,\gamma_3)$ using

$$Z(\gamma_1,\gamma_2,\gamma_3)=Y(\gamma_1,\gamma_2,\gamma_1)+Y(\gamma_2,\gamma_3,\gamma_3-p_B)\text{ with }Y(\underline{\gamma},\bar{\gamma},\Gamma)=\frac{\int_{\underline{\gamma}}^{\bar{\gamma}}\ln(\frac{\gamma-\Gamma}{\Gamma})h(\gamma)d\gamma}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma}.$$

and thus

$$\begin{aligned}\frac{dZ}{d\gamma_2}=&\left\{\frac{\partial Y(\gamma_1,\gamma_2,\gamma_1)}{\partial\underline{\gamma}}+\frac{\partial Y(\gamma_1,\gamma_2,\gamma_1)}{\partial\Gamma}\right\}\frac{d\gamma_1}{d\gamma_2}+\frac{\partial Y(\gamma_1,\gamma_2,\gamma_1)}{\partial\bar{\gamma}}+\frac{\partial Y(\gamma_2,\gamma_3,\gamma_3-p_B)}{\partial\underline{\gamma}}\\&+\left\{\frac{\partial Y(\gamma_2,\gamma_3,\gamma_3-p_B)}{\partial\bar{\gamma}}+\frac{\partial Y(\gamma_2,\gamma_3,\gamma_3-p_B)}{\partial\Gamma}\right\}(1-\frac{d\gamma_1}{d\gamma_2})\\&<0\end{aligned}$$

Now, we have that

$$\frac{\partial Y}{\partial\underline{\gamma}}=h(\underline{\gamma})\frac{\int_{\underline{\gamma}}^{\bar{\gamma}}\int_{\underline{\gamma}}^{\gamma}\frac{1}{\tilde{\gamma}-\Gamma}d\tilde{\gamma}h(\gamma)d\gamma}{(\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma)^2},\ \frac{\partial Y}{\partial\bar{\gamma}}=h(\bar{\gamma})\frac{\int_{\underline{\gamma}}^{\bar{\gamma}}\int_{\gamma}^{\bar{\gamma}}\frac{1}{\tilde{\gamma}-\Gamma}d\tilde{\gamma}h(\gamma)d\gamma}{(\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma)^2}\text{ and}$$

$$\frac{\partial Y}{\partial\Gamma}=-\frac{\int_{\underline{\gamma}}^{\bar{\gamma}}\left\{\frac{1}{\gamma-\Gamma}+\frac{1}{\Gamma}\right\}h(\gamma)d\gamma}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma}=-\frac{h(\bar{\gamma})\ln(\bar{\gamma}-\Gamma)-h(\underline{\gamma})\ln(\underline{\gamma}-\Gamma)-\int_{\underline{\gamma}}^{\bar{\gamma}}\ln(\gamma-\Gamma)h'(\gamma)d\gamma+\int_{\underline{\gamma}}^{\bar{\gamma}}\frac{1}{\Gamma}h(\gamma)d\gamma}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma}.$$

After some algrebra:

$$\frac{\partial Y}{\partial\underline{\gamma}}+\frac{\partial Y}{\partial\Gamma}=-\frac{\int_{\underline{\gamma}}^{\bar{\gamma}}\left\{\left\{\frac{h(\underline{\gamma})}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma}+\frac{h'(\gamma)}{h(\gamma)}\right\}\int_{\gamma}^{\bar{\gamma}}\frac{1}{\tilde{\gamma}-\Gamma}d\tilde{\gamma}+\frac{1}{\Gamma}\right\}h(\gamma)d\gamma}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma}\text{ and}$$

$$\frac{\partial Y}{\partial\bar{\gamma}}+\frac{\partial Y}{\partial\Gamma}=-\frac{\int_{\underline{\gamma}}^{\bar{\gamma}}\left\{\left\{\frac{h(\bar{\gamma})}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma}-\frac{h'(\gamma)}{h(\gamma)}\right\}\int_{\underline{\gamma}}^{\gamma}\frac{1}{\tilde{\gamma}-\Gamma}d\tilde{\gamma}+\frac{1}{\Gamma}\right\}h(\gamma)d\gamma}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\gamma)d\gamma}.$$

Recalling that $\frac{d\gamma_1}{d\gamma_2} = \int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma / \int_{\gamma_1}^{\gamma_3} h(\gamma)d\gamma$, the condition is

$$
\begin{aligned}
0 > \frac{dZ}{d\gamma_2} = &-\frac{\int_{\gamma_1}^{\gamma_2} \left\{ \left\{ \frac{h(\gamma_1)}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} + \frac{h'(\gamma)}{h(\gamma)} \right\} \int_{\gamma}^{\gamma_2} \frac{1}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma} + \frac{1}{\gamma_1} \right\} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} \frac{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3} h(\gamma)d\gamma} \\
&- \frac{\int_{\gamma_2}^{\gamma_3} \left\{ \left\{ \frac{h(\gamma_3)}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} - \frac{h'(\gamma)}{h(\gamma)} \right\} \int_{\gamma_2}^{\gamma} \frac{1}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma} + \frac{1}{\gamma_3-p_B} \right\} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} \frac{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_3} h(\gamma)d\gamma} \\
&+ \frac{\frac{h(\gamma_2)}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} \int_{\gamma_1}^{\gamma_2} \int_{\gamma}^{\gamma_2} \frac{1}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} + \frac{\frac{h(\gamma_2)}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} \int_{\gamma_2}^{\gamma_3} \int_{\gamma_2}^{\gamma} \frac{1}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}
\end{aligned}
$$

or, after some algebra

$$
\begin{aligned}
0 > &-\underbrace{\frac{\int_{\gamma_1}^{\gamma_2} \left\{ -\frac{\int_{\gamma_1}^{\gamma_2} h'(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} + \frac{h'(\gamma)}{h(\gamma)} \right\} \int_{\gamma}^{\gamma_2} \frac{1}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma}}_{\text{condition 1A}} \int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma - \underbrace{\frac{\int_{\gamma_2}^{\gamma_3} \left\{ \frac{\int_{\gamma_2}^{\gamma_3} h'(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} - \frac{h'(\gamma)}{h(\gamma)} \right\} \int_{\gamma_2}^{\gamma} \frac{1}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}}_{\text{condition 1B}} \int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma \\
&\underbrace{-\frac{\int_{\gamma_1}^{\gamma_2} \left\{ -\frac{h(\gamma_2)}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} \int_{\gamma}^{\gamma_2} \frac{1}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma} + \frac{1}{\gamma_1} \right\} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} \int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma - \frac{\int_{\gamma_2}^{\gamma_3} \left\{ -\frac{h(\gamma_2)}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} \int_{\gamma_2}^{\gamma} \frac{1}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma} + \frac{1}{\gamma_3-p_B} \right\} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} \int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma}_{\text{conditions 2 and 3}}
\end{aligned}
$$

- conditions 1A and 1B:

$$
-\frac{\int_{\underline{\gamma}}^{\bar{\gamma}} h'(\tilde{\gamma})d\tilde{\gamma}}{\int_{\underline{\gamma}}^{\bar{\gamma}} h(\tilde{\gamma})d\tilde{\gamma}} h(\gamma) + h'(\gamma) \text{ is monotone decreasing} \Leftrightarrow \underbrace{-\frac{\int_{\underline{\gamma}}^{\bar{\gamma}} h'(\tilde{\gamma})d\tilde{\gamma}}{\int_{\underline{\gamma}}^{\bar{\gamma}} h(\tilde{\gamma})d\tilde{\gamma}} h'(\gamma) + h''(\gamma) < 0}_{\text{condition 1}}
$$

  as $\int_{\underline{\gamma}}^{\bar{\gamma}} \left\{ -\frac{\int_{\underline{\gamma}}^{\bar{\gamma}} h'(\tilde{\gamma})d\tilde{\gamma}}{\int_{\underline{\gamma}}^{\bar{\gamma}} h(\tilde{\gamma})d\tilde{\gamma}} h(\gamma) + h'(\gamma) \right\} d\gamma = 0$ and $\int_{\gamma}^{\gamma_2} \frac{1}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma}$ is decreasing and $\int_{\gamma_2}^{\gamma} \frac{1}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma}$ is increasing, we have that

$$
\begin{aligned}
0 &< \int_{\gamma_1}^{\gamma_2} \left\{ -\frac{\int_{\gamma_1}^{\gamma_2} h'(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} h(\gamma) + h'(\gamma) \right\} \int_{\gamma}^{\gamma_2} \frac{1}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma} d\gamma \text{ and} \\
0 &< \int_{\gamma_2}^{\gamma_3} \left\{ -\left\{ -\frac{\int_{\gamma_2}^{\gamma_3} h'(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} h(\gamma) + h'(\gamma) \right\} \int_{\gamma_2}^{\gamma} \frac{1}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma} \right\} d\gamma
\end{aligned}
$$

  which are conditions 1A and 1B.

- conditions 2 and 3:

$$
\begin{aligned}
0 &> -\frac{\int_{\gamma_1}^{\gamma_2} \left\{ -\frac{\int_{\gamma}^{\gamma_2} \frac{h(\gamma_2)}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma}}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} + \int_{\gamma_2}^{\gamma_3} \frac{h(\gamma)}{\gamma_1} d\gamma \right\} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} - \frac{\int_{\gamma_2}^{\gamma_3} \left\{ -\frac{\int_{\gamma_2}^{\gamma} \frac{h(\gamma_2)}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma}}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} + \int_{\gamma_1}^{\gamma_2} \frac{h(\gamma)}{\gamma_3-p_B} d\gamma \right\} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} \Leftrightarrow \\
0 &> h(\gamma_2) \frac{\int_{\gamma_1}^{\gamma_2} \int_{\gamma}^{\gamma_2} \frac{1}{\tilde{\gamma}-\gamma_1} d\tilde{\gamma} h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} - \frac{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}{\gamma_1} + h(\gamma_2) \frac{\int_{\gamma_2}^{\gamma_3} \int_{\gamma_2}^{\gamma} \frac{1}{\tilde{\gamma}-(\gamma_3-p_B)} d\tilde{\gamma} h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} - \frac{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma}{\gamma_3-p_B} \Leftrightarrow \\
0 &> h(\gamma_2) \frac{\int_{\gamma_1}^{\gamma_2} \ln(\frac{\gamma_2-\gamma_1}{\gamma-\gamma_1}) h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma} - \frac{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma}{\gamma_1} + h(\gamma_2) \frac{\int_{\gamma_2}^{\gamma_3} \ln(\frac{\gamma-(\gamma_3-p_B)}{\gamma_2-(\gamma_3-p_B)}) h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3} h(\gamma)d\gamma} - \frac{\int_{\gamma_1}^{\gamma_2} h(\gamma)d\gamma}{\gamma_3-p_B} \Leftrightarrow
\end{aligned}
$$

$$0>h(\gamma_2)\frac{\int_{\gamma_1}^{\gamma_2}\ln(\frac{\gamma_2-\gamma_1}{\gamma-\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}-\frac{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{\gamma_2-\gamma_1}+h(\gamma_2)\frac{\int_{\gamma_2}^{\gamma_3}\ln(\frac{\gamma-(\gamma_3-p_B)}{\gamma_2-(\gamma_3-p_B)})h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}-\frac{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}{\gamma_2-(\gamma_3-p_B)}\Leftrightarrow$$

$$0>\underbrace{\frac{\int_{\gamma_1}^{\gamma_2}\ln(\frac{\gamma_2-\gamma_1}{\gamma-\gamma_1})h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}-\frac{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{h(\gamma_2)(\gamma_2-\gamma_1)}}_{<0,\text{ condition 2}}+\underbrace{\frac{\int_{\gamma_2}^{\gamma_3}\ln(\frac{\gamma-(\gamma_3-p_B)}{\gamma_2-(\gamma_3-p_B)})h(\gamma)d\gamma}{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}-\frac{\int_{\gamma_2}^{\gamma_3}h(\gamma)d\gamma}{h(\gamma_2)(\gamma_2-(\gamma_3-p_B))}}_{<0,\text{ condition 3}}$$

Thus, conditions 1A, 1B, 2 and 3 reduce to:

$$-\frac{\int_{\underline{\gamma}}^{\bar{\gamma}}h'(\tilde{\gamma})d\tilde{\gamma}}{\int_{\underline{\gamma}}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}}h'(\gamma)+h''(\gamma)<0 \text{ for } \gamma\in[\underline{\gamma},\bar{\gamma}]$$

$$\frac{\int_{\underline{\gamma}}^{\gamma}\ln(\frac{\gamma-\underline{\gamma}}{\tilde{\gamma}-\underline{\gamma}})h(\tilde{\gamma})d\tilde{\gamma}}{\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}}-\frac{\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}}{h(\gamma)(\gamma-\underline{\gamma})}<0 \text{ for } \gamma\in[\underline{\gamma},\bar{\gamma}]$$

$$\frac{\int_{\gamma}^{\bar{\gamma}}\ln(\frac{\tilde{\gamma}-(\bar{\gamma}-p_B)}{\gamma-(\bar{\gamma}-p_B)})h(\tilde{\gamma})d\tilde{\gamma}}{\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}}-\frac{\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}}{h(\gamma)(\gamma-(\bar{\gamma}-p_B))}<0 \text{ for } \gamma\in[\underline{\gamma},\bar{\gamma}] \text{ and } \underline{\gamma}\geq\bar{\gamma}-p_B$$

It is easy two see that when $h(\gamma)$ is decreasing and concave, the above conditions are satisfied:

1. conditions 1A and 1B are obvious as the signs of both terms are negative and $-\int_{\underline{\gamma}}^{\bar{\gamma}}h'(\tilde{\gamma})d\tilde{\gamma}/\int_{\underline{\gamma}}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}>0$ when $h(\gamma)$ is concave decreasing.
2. condition 2 can be seen as follows:

$$\frac{h(\underline{\gamma})(\gamma-\underline{\gamma})}{\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}}-\frac{\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}}{h(\gamma)(\gamma-\underline{\gamma})}<0\Rightarrow\frac{\int_{\underline{\gamma}}^{\gamma}\ln(\frac{\gamma-\underline{\gamma}}{\tilde{\gamma}-\underline{\gamma}})h(\tilde{\gamma})d\tilde{\gamma}}{\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}}-\frac{\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}}{h(\gamma)(\gamma-\underline{\gamma})}<0$$

and

$$\sqrt{h(\underline{\gamma})h(\gamma)}(\gamma-\underline{\gamma})<\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}$$

and as $\sqrt{ab}<\frac{a+b}{2}\Leftrightarrow 4ab<(a+b)^2\Leftrightarrow 0<(a-b)^2$ which is always true when $a\neq b$, we have that

$$h''(\gamma)<0\Rightarrow\sqrt{h(\underline{\gamma})h(\gamma)}(\gamma-\underline{\gamma})<\frac{h(\underline{\gamma})+h(\gamma)}{2}(\gamma-\underline{\gamma})<\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}$$

(as $\frac{h(\underline{\gamma})+h(\gamma)}{2}(\gamma-\underline{\gamma})$ is the area under the trapezoid that is less than the area under $h(\tilde{\gamma})$).

3. condition 3 can be seen as follows: With $\bar{\gamma}-p_B=\gamma-\varepsilon$, we rewrite:

$$\frac{\int_{\gamma}^{\bar{\gamma}}\ln(\frac{\tilde{\gamma}-(\bar{\gamma}-p_B)}{\gamma-(\bar{\gamma}-p_B)})h(\tilde{\gamma})d\tilde{\gamma}}{\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}}-\frac{\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}}{h(\gamma)(\gamma-(\bar{\gamma}-p_B))}<0\Leftrightarrow$$

$$\frac{\int_{\gamma}^{\bar{\gamma}}\ln(\frac{\tilde{\gamma}-(\gamma-\varepsilon)}{\gamma-(\gamma-\varepsilon)})h(\tilde{\gamma})d\tilde{\gamma}}{\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}}-\frac{\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}}{h(\gamma)(\gamma-(\gamma-\varepsilon))}<0\Leftrightarrow$$

$$h(\gamma)\varepsilon\int_{\gamma}^{\bar{\gamma}}\ln(\frac{\tilde{\gamma}-(\gamma-\varepsilon)}{\gamma-(\gamma-\varepsilon)})h(\tilde{\gamma})d\tilde{\gamma}-(\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma})^2<0$$

We show that the right hand side is decreasing in $\bar{\gamma}$ (and is zero for $\bar{\gamma}=\gamma$):

$$\frac{d}{d\bar{\gamma}}\left\{h(\gamma)\varepsilon\int_{\gamma}^{\bar{\gamma}}\ln(\frac{\tilde{\gamma}-(\gamma-\varepsilon)}{\gamma-(\gamma-\varepsilon)})h(\tilde{\gamma})d\tilde{\gamma}-(\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma})^2\right\}$$

$$=h(\gamma)\varepsilon\ln(\frac{\bar{\gamma}-(\gamma-\varepsilon)}{\gamma-(\gamma-\varepsilon)})h(\bar{\gamma})-2h(\bar{\gamma})\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}<0\Leftrightarrow$$

$$0>h(\gamma)\varepsilon\ln(\frac{\bar{\gamma}-\gamma+\varepsilon}{\varepsilon})-2\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}$$

And as $\ln(1+a)\leq a$, we have

$$h(\gamma)\varepsilon\frac{\bar{\gamma}-\gamma}{\varepsilon}-2\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}<0\Rightarrow h(\gamma)\varepsilon\ln(\frac{\bar{\gamma}-\gamma+\varepsilon}{\varepsilon})-2\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}<0$$

and thus

$$h''(\gamma)<0\Rightarrow\frac{h(\underline{\gamma})+h(\gamma)}{2}(\gamma-\underline{\gamma})<\int_{\underline{\gamma}}^{\gamma}h(\tilde{\gamma})d\tilde{\gamma}\Rightarrow\frac{h(\gamma)}{2}(\bar{\gamma}-\gamma)<$$

$$\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}\Rightarrow h(\gamma)\varepsilon\ln(\frac{\bar{\gamma}-\gamma+\varepsilon}{\varepsilon})-2\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma}<0$$

and thus $h(\gamma)\varepsilon\int_{\gamma}^{\bar{\gamma}}\ln(\frac{\tilde{\gamma}-(\gamma-\varepsilon)}{\gamma-(\gamma-\varepsilon)})h(\tilde{\gamma})d\tilde{\gamma}-(\int_{\gamma}^{\bar{\gamma}}h(\tilde{\gamma})d\tilde{\gamma})^2$ is decreasing in $\bar{\gamma}$, starting from 0 for $\bar{\gamma}=\gamma$, which implies that it is negative.

□

Proof of Lemma 9: We have:

$$v_A(\gamma_1,\gamma_2)=1+\int_{\gamma_2}^{1}\frac{\gamma-\gamma_2+\gamma_1}{p_A}h(\gamma)d\gamma+\int_{\gamma_1}^{\gamma_2}\frac{\gamma_1}{p_A}h(\gamma)d\gamma+\int_{0}^{\gamma_1}\frac{\gamma}{p_A}h(\gamma)d\gamma$$

we can rewrite the objective function as

$$\begin{aligned}L(\gamma_1,\gamma_2)=&p_A+\int_{\gamma_2}^{1}(\gamma-\gamma_2+\gamma_1)h(\gamma)d\gamma+\int_{\gamma_1}^{\gamma_2}\gamma_1 h(\gamma)d\gamma+\int_{0}^{\gamma_1}\gamma h(\gamma)d\gamma\\&+p_A\{\int_{\gamma_2}^{1}T(\frac{\gamma-\gamma_2+\gamma_1}{p_A})h(\gamma)d\gamma+\int_{\gamma_1}^{\gamma_2}T(\frac{\gamma_1}{p_A})h(\gamma)d\gamma+\int_{0}^{\gamma_1}T(\frac{\gamma}{p_A})h(\gamma)d\gamma\}\end{aligned}$$

$$+(1-p_A)\{\int_{\gamma_1}^{\gamma_2}T(\frac{\gamma-\gamma_1}{1-p_A})h(\gamma)d\gamma\}$$

and the conditions are: $p_B=\gamma_2-\gamma_1\Rightarrow 0=\frac{d\gamma_1}{d\gamma_2}=1$. We obtain via straightforward derivation:

$$\frac{d\hat{L}}{d\gamma_2}=\frac{\partial L}{\partial\gamma_1}\frac{d\gamma_1}{d\gamma_2}+\frac{\partial L}{\partial\gamma_2}=\int_{\gamma_1}^{\gamma_2}\left\{1-\ln(\frac{\gamma_1}{\gamma-\gamma_1})+\ln(\frac{p_A}{1-p_A})\right\}h(\gamma)d\gamma$$

Hence, the first order condition is $Z(\gamma_1,\gamma_2)=0$ (and $p_B=\gamma_2-\gamma_1$), where

$$Z(\gamma_1,\gamma_2)=\int_{\gamma_1}^{\gamma_2}\left\{1-\ln(\frac{\gamma_1}{\gamma-\gamma_1})+\ln(\frac{p_A}{1-p_A})\right\}h(\gamma)d\gamma$$

When $Z=0$, the second derivative is $\frac{d^2\hat{L}}{d\gamma_2^2}=\frac{dZ}{d\gamma_1}=\frac{\partial Z}{\partial\gamma_1}\frac{d\gamma_1}{d\gamma_2}+\frac{\partial Z}{\partial\gamma_2}$ and thus

$$\begin{aligned}\frac{dZ}{d\gamma_1} = -\int_{\gamma_1}^{\gamma_2}\left\{\frac{1}{\gamma_1}+\frac{1}{\gamma-\gamma_1}\right\}h(\gamma)d\gamma - \left\{1-\ln(\frac{\gamma_1}{\gamma_1-\gamma_1})+\ln(\frac{p_A}{1-p_A})\right\}h(\gamma_1)\\ +\left\{1-\ln(\frac{\gamma_1}{\gamma_2-\gamma_1})+\ln(\frac{p_A}{1-p_A})\right\}h(\gamma_2)\end{aligned}$$

with

$$\int_{\gamma_1}^{\gamma_2}\frac{1}{\tilde{\gamma}-\gamma_1}h(\tilde{\gamma})d\tilde{\gamma}=h(\gamma_1)\ln(\frac{\gamma_2-\gamma_1}{\gamma_1-\gamma_1})-\int_{\gamma_1}^{\gamma_2}\ln(\frac{\tilde{\gamma}-\gamma_1}{\gamma_2-\gamma_1})h'(\tilde{\gamma})d\tilde{\gamma}$$

we have

$$\frac{dZ}{d\gamma_1}=-\frac{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{\gamma_1}+\int_{\gamma_1}^{\gamma_2}\ln(\frac{\tilde{\gamma}-\gamma_1}{\gamma_2-\gamma_1})h'(\tilde{\gamma})d\tilde{\gamma}+\left\{1-\ln(\frac{\gamma_1}{\gamma_2-\gamma_1})+\ln(\frac{p_A}{1-p_A})\right\}(h(\gamma_2)-h(\gamma_1))$$

As

$$0=Z(\gamma_1,\gamma_2)\Rightarrow\ln(\frac{p_A}{1-p_A})=-\frac{\int_{\gamma_1}^{\gamma_2}\left\{1-\ln(\frac{\gamma_1}{\gamma-\gamma_1})\right\}h(\gamma)d\gamma}{\int_{\gamma_1}^{\gamma_2}h(\tilde{\gamma})d\tilde{\gamma}}$$

we have that

$$\begin{aligned}\frac{dZ}{d\gamma_1}=-\frac{\int_{\gamma_1}^{\gamma_2}h(\gamma)d\gamma}{\gamma_1}+\int_{\gamma_1}^{\gamma_2}\ln(\frac{\tilde{\gamma}-\gamma_1}{\gamma_2-\gamma_1})h'(\tilde{\gamma})d\tilde{\gamma}+\int_{\gamma_1}^{\gamma_2}\left\{1-\ln(\frac{\gamma_1}{\gamma_2-\gamma_1})\right\}h'(\gamma)d\gamma\\ -\int_{\gamma_1}^{\gamma_2}\left\{1-\ln(\frac{\gamma_1}{\gamma-\gamma_1})\right\}h(\gamma)d\gamma\frac{\int_{\gamma_1}^{\gamma_2}h'(\tilde{\gamma})d\tilde{\gamma}}{\int_{\gamma_1}^{\gamma_2}h(\tilde{\gamma})d\tilde{\gamma}}\end{aligned}$$

and after some algebraic manipulation:

$$\begin{aligned}\frac{dZ}{d\gamma_1}=\int_{\gamma_1}^{\gamma_2}\left\{-\frac{1}{\gamma_1}+\left\{\frac{h'(\gamma)}{h(\gamma)}-\frac{\int_{\gamma_1}^{\gamma_2}h'(\tilde{\gamma})d\tilde{\gamma}}{\int_{\gamma_1}^{\gamma_2}h(\tilde{\gamma})d\tilde{\gamma}}\right\}\ln(\gamma-\gamma_1)\right\}h(\gamma)d\gamma<0\Leftrightarrow\\ 0>-\frac{1}{\gamma_1}+\int_{\gamma_1}^{\gamma_2}\left\{\frac{h'(\gamma)}{h(\gamma)}\int_{\gamma_1}^{\gamma_2}h(\tilde{\gamma})d\tilde{\gamma}-(h(\gamma_2)-h(\gamma_1))\right\}\ln(\gamma-\gamma_1)h(\gamma)d\gamma\frac{1}{(\int_{\gamma_1}^{\gamma_2}h(\tilde{\gamma})d\tilde{\gamma})^2}\end{aligned}$$

and with $\gamma_1=\underline{\gamma}$, $\Delta=0$ and $\gamma_2=\bar{\gamma}$, we obtain the same expression of Equation (13) for which a sufficient condition is that $h(\gamma)$ is concave.

□

Proof of Lemma 10 Via straightforward derivation, we have:

$$\begin{aligned}h'(\gamma)=\frac{d}{d\gamma}\left\{\frac{1}{g(\bar{G}^{-1}(\gamma))}\right\}=\frac{g'(\bar{G}^{-1}(\gamma))}{(g(\bar{G}^{-1}(\gamma)))^2}\frac{1}{g(\bar{G}^{-1}(\gamma))}\text{ and}\\ h''(\gamma)=\frac{d^2}{d\gamma^2}\left\{\frac{1}{g(\bar{G}^{-1}(\gamma))}\right\}=\frac{\frac{g''(\bar{G}^{-1}(\gamma))}{g(\bar{G}^{-1}(\gamma))}(g(\bar{G}^{-1}(\gamma)))^3-g'(\bar{G}^{-1}(\gamma))3(g(\bar{G}^{-1}(\gamma)))^2\frac{g'(\bar{G}^{-1}(\gamma))}{g(\bar{G}^{-1}(\gamma))}}{(g(\bar{G}^{-1}(\gamma)))^6}\\ =\frac{g''(\bar{G}^{-1}(\gamma))g(\bar{G}^{-1}(\gamma))-3(g'(\bar{G}^{-1}(\gamma)))^2}{(g(\bar{G}^{-1}(\gamma)))^5}\end{aligned}$$

and thus $h(\gamma)$ is decreasing concave when $g(\omega)$ is decreasing concave too.

□

4. Optimal information structures for $\bar{\omega}=4$

Figures 14 - 19 show the optimal information structures when $\bar{\omega}=4$ and there is a single announcement at $t=0$. As shown in Figure 13, each possible information structure is assigned a color, with ordinal information structures (e.g., ABBB, AABB) various shades of green and onion structures (e.g., ABAA, AABA) various shades of red. The structure ABAB is neither ordinal nor onion and is assigned a gray color.

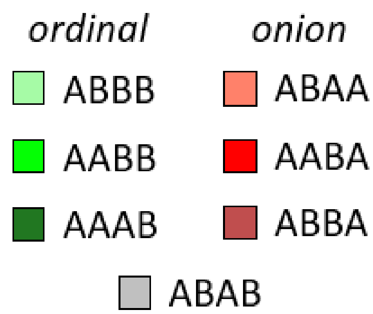


Figure 13 **Legend for optimal information structures for $\bar{\omega}=4$ in Figures 14 - 19 (color figure can be viewed in on-line document)**

In Figures 14 - 19 the x-axis is g_2^{-1} and the y-axis is g_3^{-1}. The value of g_1^{-1} is constant for each Figure (e.g., in Figure 14, $g_1^{-1}=0.01$) and therefore $g_4^{-1}=1-g_1^{-1}-g_2^{-1}-g_3^{-1}$ is determined as well. The six plots within each figure show how the optimal structures change as β varies, from $\beta=0$ for loss averse customers to $\beta=10,000$ for essentially risk-conscious customers.

Examining the pattern within each Figure, onion information structures are dominant when customers are loss-averse and ordinal structures are dominant when customers are risk conscious - a pattern we also saw for $\bar{\omega}=3$ in Section 4.1 and the numerical experiments in Section 5, Looking across Figures, given a low value of β, AABA tends to be optimal for low values of g_1^{-1}, ABBA for medium values of g_1^{-1}, and ABAA for high values of g_1^{-1}. Likewise, when β is large, AAAB/AABB/ABBB are optimal for low/medium/high values of g_1^{-1}. The latter pattern is intuitive: to minimize the variance of the posterior distribution, provide the most precise outcomes for the most likely delay durations.

That said, any of the seven possible information structures may be optimal for particular values of the prior distribution and β, and the optimality patterns across the prior distributions are complex.

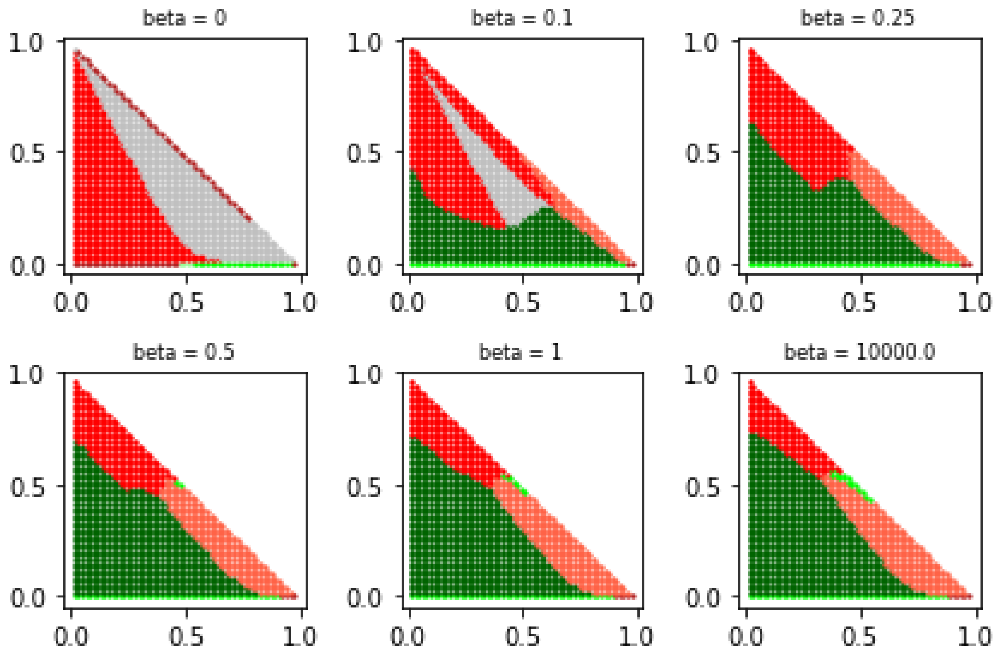


Figure 14 Optimal Structure for $\bar{\omega} = 4$, $g_1 = 0.01$ (color figure can be viewed in on-line document)

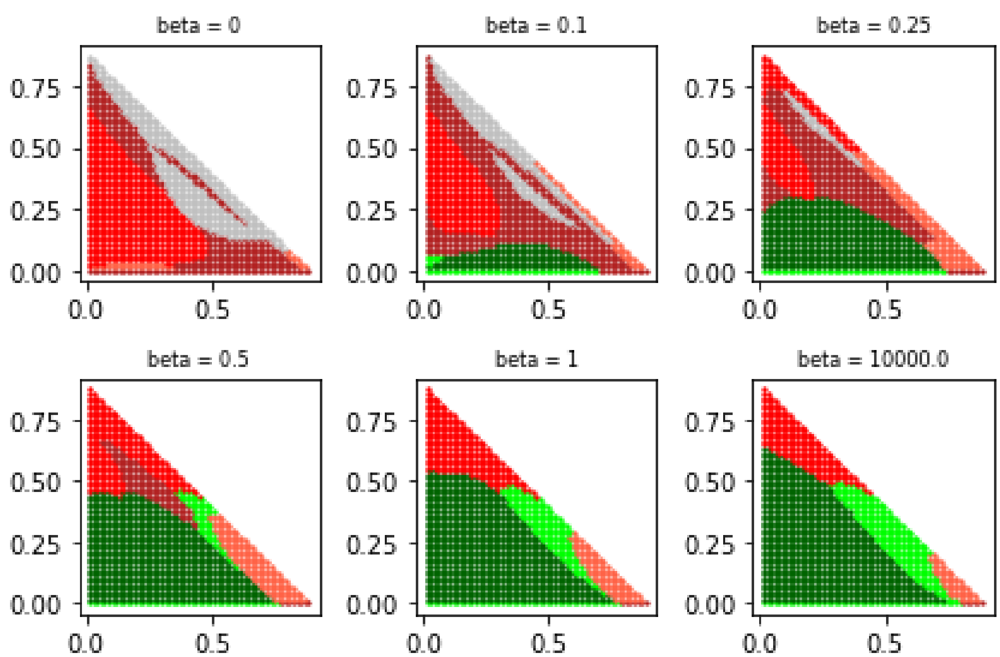


Figure 15 Optimal Structure for $\bar{\omega} = 4$, $g_1 = 0.1$

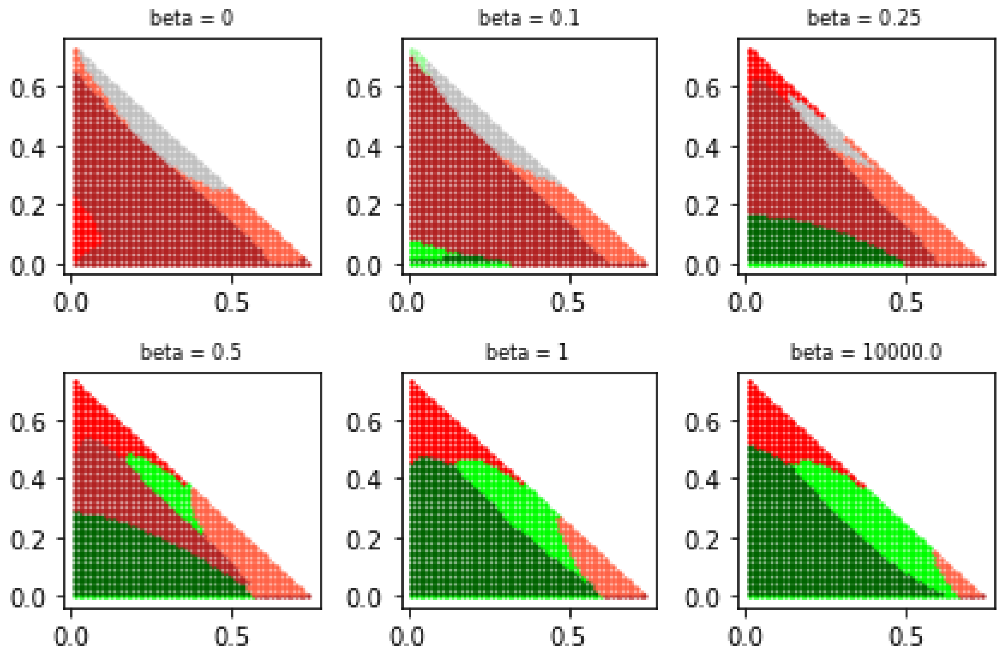


Figure 16 **Optimal Structure for** $\bar{\omega} = 4,\ g_1 = 0.25$

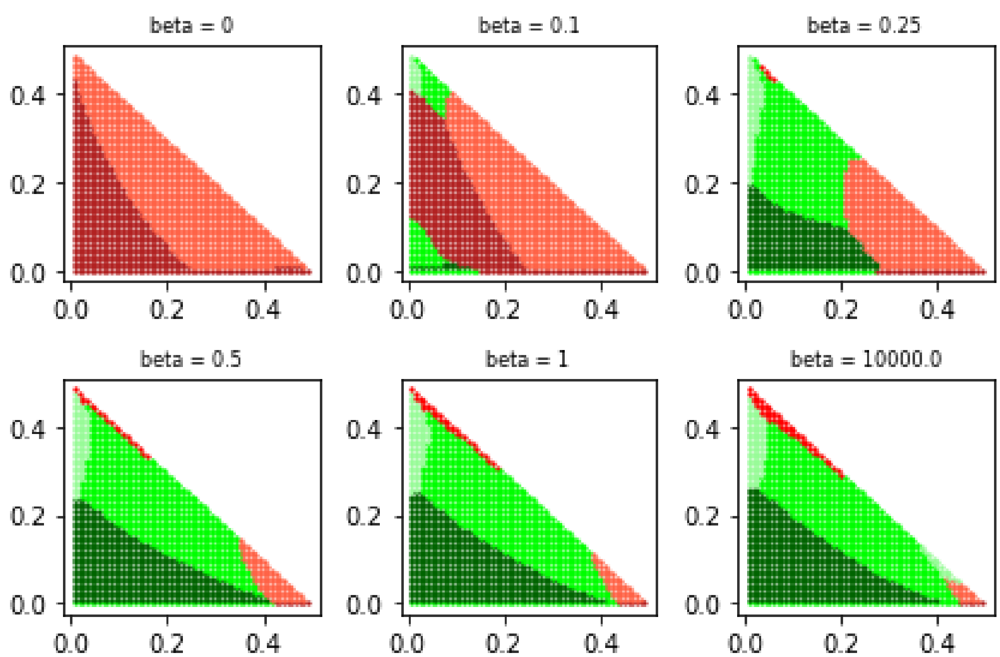


Figure 17 **Optimal Structure for** $\bar{\omega} = 4,\ g_1 = 0.5$

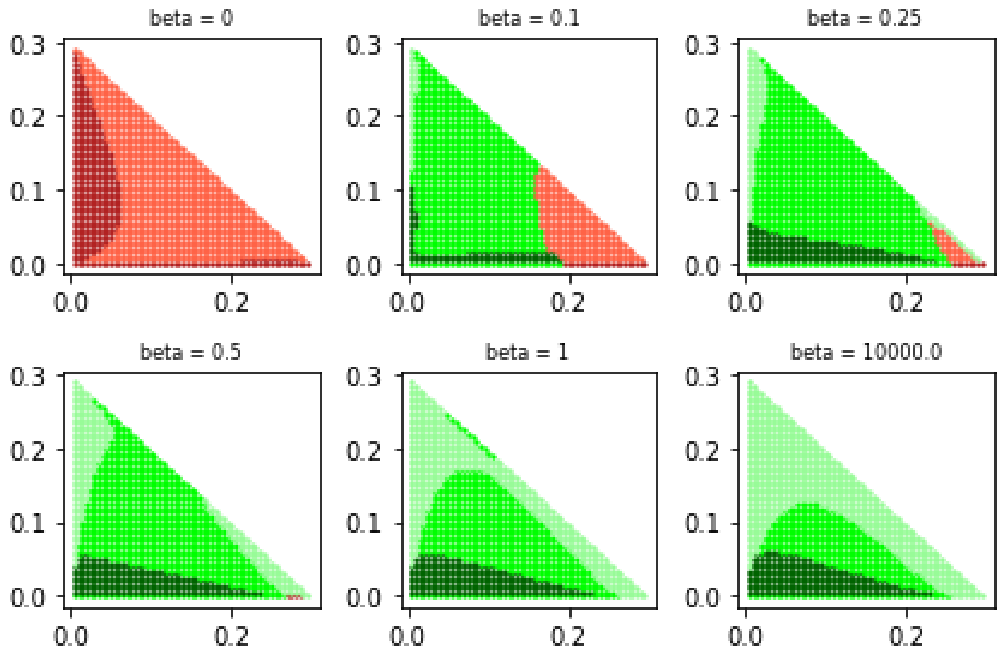


Figure 18 **Optimal Structure for** $\bar{\omega} = 4,\ g_1 = 0.7$

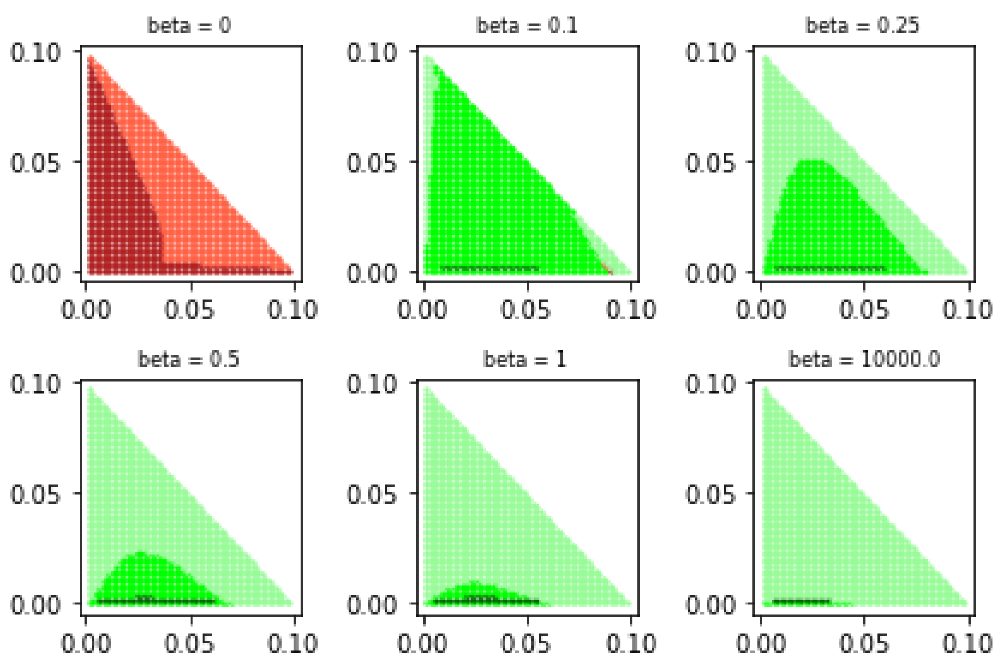


Figure 19 **Optimal Structure for** $\bar{\omega} = 4,\ g_1 = 0.9$