Questions

1. **[15 marks]** *Digital Constellations*

   A digital modulation scheme consisting of the following four signals is used to communication information over an AWGN channel with power spectral density $N_0/2 = 1$.

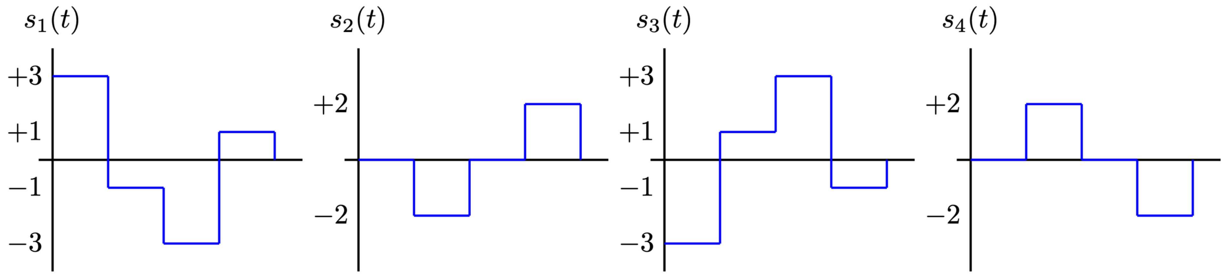

Figure 1: 4-ary modulation signal set

   Bits arrive at the digital modulator with an unequal distribution, with bit b_k being '0' with probability 0.6 and '1' with probability 0.4.

   (a) (2 marks) Determine a set of orthonormal basis functions for this modulation scheme and provide the corresponding signal constellation diagram.

   (b) (2 marks) Choose a mapping of bits to constellation points that will minimise the average energy per symbol, E_S, while also giving decent bit error performance. Please explain your thinking in determining this mapping.

   (c) (2 marks) Calculate E_S based on the mapping you chose in (b).

   (d) (4 marks) Determine the MAP decision boundaries for deciding signal $s_1(t)$ is sent and clearly mark these on your constellation diagram.

   (e) (4 marks) Use the MAP decision boundaries you found in (d) along with the union bound to upper bound the symbol error probability with signal $s_1(t)$ is sent, i.e., upper bound $P(\text{error}|s = s_1)$.

   Please list all of the resources you used to answer this question including references to the subject notes, workshops and problem sets as well as any websites or textbooks etc.

2. **[15 marks]** *Communication Design*

   Consider the baseband 2-PAM communication system of Figure 2 that employs a root raised cosine (RRC) pulse at both the transmit and receive filters. You are to choose parameters for this system to maximise the bit rate under the following constraints.

   - Maximum available bandwidth: 1 MHz
   - Maximum transmit power: 3 W
   - Maximum allowable bit error rate: 10^{-3}

The transmitted signal passes through a benign channel, but the sampling error in the symbol sampling can be up to 20% of the chosen symbol period, T. This worst case symbol timing error is modelled as a time advance in the channel, $c(t) = \delta(t + \tau)$, where $\tau = 0.2T$.

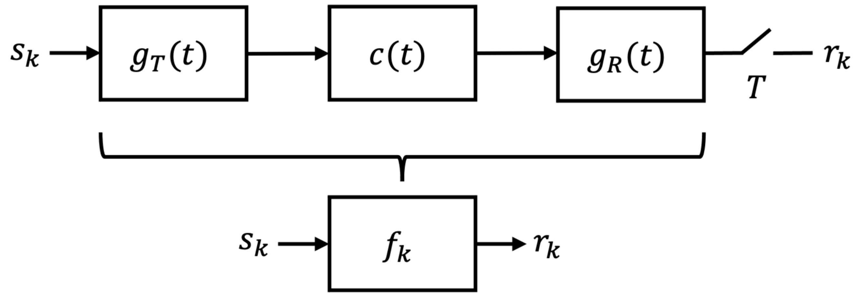

Figure 2: ISI model for 2-PAM receiver.

(a) (2 marks) For now assume that an RRC pulse with $\beta = 0.2$ is employed. Determine an equivalent discrete-time ISI model (i.e. determine FIR filter taps f_k) to capture the combined effects of $g_T(t)$, $c(t)$, $g_T(t)$ and the sampling operation. Keep only the five most significant (largest magnitude) taps in your model.

(b) (3 marks) Assume that the received symbols are also corrupted by i.i.d. Gaussian noise,

$$y_k = r_k + \eta_k,$$

where $\eta_k \sim \mathcal{N}(0, N_0/2)$. Determine an expression to calculate the bit error probability, P_b, under this ISI model when symbol-by-symbol detection is used (no equalisation) and the decision boundary is set at $y_k = 0$.

(c) (3 marks) Write a MATLAB script that repeats the previous two steps for ten evenly spaced values of $\beta \in [0, 1]$. Provide a combined plot of P_b versus E_b/N_0 for each β, where $E_b/N_0 \in [0, 15]$ dB. Include on your plot a curve for the theoretical performance of binary PAM.

(d) (3 marks) Based on your plots of the previous question, choose values of β and T to maximise the data rate subject to the original constraints of the problem and assuming $N_0/2 = 10^{-7}$ W/Hz.

Please list all of the resources you used to answer this question including references to the subject notes, workshops and problem sets as well as any websites or textbooks etc.

3. **[15 marks]** *FSK and Non-Coherent Receivers*

Consider a system using binary Frequency Shift-Keying (FSK) to send information over an AWGN channel. On the symbol interval $[0, T]$, the possible transmitted symbols are: $s_1(t) = \sqrt{E_b}\phi_1(t)$ and $s_2(t) = \sqrt{E_b}\phi_2(t)$ where

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t), \qquad \phi_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)$$

with $T = 2\,\text{ms}$ and $f_1 = f_c - f_\Delta$, $f_2 = f_c + f_\Delta$, $f_c = 5\,\text{MHz}$, $f_\Delta = 250\,\text{Hz}$.

(a) Prove that $\phi_1(t)$ and $\phi_2(t)$ are orthonormal.

(b) Assume the received signal is $r(t) = \sqrt{\frac{2E_b}{T}}\cos(2\pi f_m t) + n(t)$ where $m \in \{1, 2\}$ and $n(t)$ is white Gaussian noise with power spectral density $\frac{N_0}{2}$. The optimal receiver first correlates the received signal with $\phi_1(t)$ and $\phi_2(t)$ to produce $r_1 = \langle r(t), \phi_1(t)\rangle$ and $r_2 = \langle r(t), \phi_2(t)\rangle$. The detector then selects s_1 if $r_1 > r_2$ otherwise it selects s_2.

Determine the conditional joint pdf of r_1 and r_2 given s_m was sent, $p(r_1, r_2 \,|\, s_m)$, $m \in \{1, 2\}$ and then determine the bit error probability as a function of E_b/N_0.

(c) Now suppose that the received signal is $r(t) = \sqrt{\frac{2E_b}{T}}\cos(2\pi f_m t + \theta) + n(t)$ for some unknown θ. Assuming the same receiver as in (b), determine an expression for the conditional joint pdf of r_1 and r_2 given s_m was sent, $p(r_1, r_2 \,|\, s_m)$, $m \in \{1, 2\}$ and find the error probability as a function of E_b/N_0 and θ. For $E_b/N_0 = 7$ dB, plot the bit error probability versus θ for $0 \le \theta \le 2\pi$.

(d) An alternative (noncoherent) receiver operates by producing two correlator outputs for each frequency:

$$r_{1c} = \sqrt{\frac{2}{T}}\int_0^T r(t)\cos(2\pi f_1 t)\,dt, \qquad r_{1s} = \sqrt{\frac{2}{T}}\int_0^T r(t)\sin(2\pi f_1 t)\,dt$$

and

$$r_{2c} = \sqrt{\frac{2}{T}}\int_0^T r(t)\cos(2\pi f_2 t)\,dt, \qquad r_{2s} = \sqrt{\frac{2}{T}}\int_0^T r(t)\sin(2\pi f_2 t)\,dt.$$

The detector then forms $r_1^2 = r_{1c}^2 + r_{1s}^2$ and $r_2^2 = r_{2c}^2 + r_{2s}^2$ and picks s_1 if $r_1^2 > r_2^2$ and s_2 otherwise.

If the input signal is $r(t) = \sqrt{\frac{2E_b}{T}}\cos(2\pi f_m t + \theta) + n(t)$ then it turns out that if s_1 is sent:

$$\begin{aligned} r_{1c} &= \sqrt{E_b}\cos(\theta) + n_{1c} \\ r_{1s} &= \sqrt{E_b}\sin(\theta) + n_{1s} \\ r_{2c} &= n_{2c} \\ r_{2s} &= n_{2s} \end{aligned}$$

where n_{1c}, n_{1s}, n_{2c} and n_{2s} are independent and identically distributed Gaussian random variables with mean 0 and variance $N_0/2$.

It is difficult to analytically determine the exact error probability, so instead, you should calculate the error probability using a Monte Carlo simulation in Matlab.For the simulation, set $E_b/N_0 = 7$ dB (set $N_0 = 1$ and $E_b = 5$) and find the bit error probability for θ from 0 to 2π in steps of 0.1.

Compare the result to that obtained in (c). Why is this receiver "robust" to the value of θ?

Please list all of the resources you used to answer this question including references to the subject notes, workshops and problem sets as well as any websites or textbooks etc.

4. **[15 marks]** *Bit Error Rate Calculation with Random Channel Gain*

The sampled output of a matched-filter is given by

$$r = h\,x + n$$

where the transmitted symbol $x = \pm\sqrt{E_b}$ and n is a Gaussian random variable with mean 0 and variance $N_0/2$. The channel gain h is known at the receiver and $h \geq 0$. Assume the input symbols are equally likely.

(a) Explain the operation of the maximum-a-posteriori (MAP) detector for detecting whether x is $-\sqrt{E_b}$ or $\sqrt{E_b}$.

(b) Determine an expression for the bit error rate (BER) of the MAP detector in terms of h, E_b and N_0.

(c) Now suppose that h is a discrete random variable that takes values 0.25, 1 and 1.75 with respective probabilities 0.3, 0.4 and 0.3. Determine an expression for the average BER by taking the expected value of the BER expression you obtained in (b).

(d) Plot the average BER from (c) versus E_b/N_0 in the range from 0 dB to 20 dB. What happens to this BER as E_b/N_0 gets large? Can you explain this observation?

(e) Validate your results from (c) by running a Monte-Carlo simulation in MATLAB and plotting the simulated results on top of your plot from (d). This will involve repeatedly generating random realisations of x, n and h, forming r, doing the detection and then checking if there is an error.

(f) **(Harder)** Now suppose that h is a continuous random variable with pdf $p(x) = 2x\exp(-x^2)$ for $x \geq 0$. Again determine an expression for the average BER by taking the expected value of the BER expression in (b). Your initial expression for the average BER might involve an integral of a Q-function. This could be evaluated numerically, but if you work a bit harder you might actually be able to reduce this to a fairly simple closed-form expression. (Hint: Write the Q-function in its integral form and then carefully swap the order of integration.)

(g) Plot the average BER from (f) versus E_b/N_0 in the range from 0 dB to 20 dB. What happens to this BER as E_b/N_0 gets large?

Please list all of the resources you used to answer this question including references to the subject notes, workshops and problem sets as well as any websites or textbooks etc.

5. **[10 marks]** *Bandlimited Systems and Nyquist Criterion*

Consider a baseband On-Off keying (OOK) system in which the modulator is defined by a symbol interval T and a transmit pulse $p(t)$, the channel is defined by a filter $h(t)$, and the receiver is defined by a filter $q(t)$ which is sampled every T seconds. The received waveform, after the receive filter $q(t)$, is then given by $r(t) = \sum_j u_j g(t - jT)$ where $g(t) = p(t) * h(t) * q(t)$ and $\{u_j\}$ is the transmitted data sequence.

(a) What property must $g(t)$ have so that $r(kT) = u_k$ for all k and for all choices of input $\{u_j\}$? What is the Nyquist criterion for $G(f)$?

(b) Now assume that $T = 1/2$ and that $p(t)$, $h(t)$ and $q(t)$ and all their Fourier transforms are restricted to be real. Assume further that $P(f)$ and $H(f)$ are given by

$$P(f) = \begin{cases} 1, & |f| \leq 1.8 \\ 0, & |f| > 1.8 \end{cases}$$

$$H(f) = \begin{cases} |f|, & |f| \leq 1 \\ 1, & 1 \leq |f| \leq 1.7 \\ 0, & \text{otherwise} \end{cases}$$

If we choose $G(f)$ to be the raised cosine spectrum, what is the largest possible roll-off factor β. With that β, what is the suitable $Q(f)$?

(c) Redo question b) with the modification that now $H(f) = |f|$ for $|f| \leq 0.9$ and $H(f) = 0$ for $|f| > 0.9$.

(d) Now consider an OOK bandpass system with the carrier frequency $f_c = 8 \times 10^3$ Hz. Assume this is a digital system and the sampling rate is $f_s = 2 \times 10^4$ Hz, what is the bandwidth of a signal that this system can process without distortion? Then what are the maximum symbol rate that the OOK bandpass system can have?

Please list all of the resources you used to answer this question including references to the subject notes, workshops and problem sets as well as any websites or textbooks etc.

6. **[15 marks] (Harder)** *Multiuser Detection*

Consider a multiuser communication system where K-users communicate simultaneously over an additive white Gaussian channel. The kth user is assigned a finite energy signature waveform, $s_k(t), t \in [0, T]$, and it transmits data by modulating that waveform antipodally (sending $\pm s_k(t)$). The receiver observes

$$r(t) = \sum_{k=1}^{K} b_k\, s_k(t) + n(t), \qquad t \in [0, T],$$

where $n(t)$ is a realization of a white Gaussian process with power spectral density $N_0/2$, $b_k \in \{-1, 1\}$ is the k-th user's data symbol, $s_k(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_k t)$, and f_k is the carrier frequency of the k-th user. Let $T = 10^{-2}$ s and $f_k = (5 + a(k-1))\,\frac{1}{T}$, where a is a constant.

Let y_k be the output of the demodulator of the k-th user:

$$y_k = \int_0^T r(t) s_k(t) \mathrm{d}t. \qquad (1)$$

If we write $\mathbf{y} = [y_1, y_2, \ldots, y_K]^T$ and $\mathbf{b} = [b_1, b_2, \ldots, b_K]^T$, then we have

$$\mathbf{y} = \mathbf{H}\mathbf{b} + \mathbf{n}, \tag{2}$$

where $\mathbf{H}$ is the crosscorrelation matrix.

$$\mathbf{H}_{ij} = \int_0^T s_i(t)s_j(t)\mathrm{d}t, \tag{3}$$

and $\mathbf{n}$ is a zero-mean Gaussian vector with covariance matrix equal to $\frac{N_0}{2}\mathbf{H}$.

(a) If $K = 1$ show that $\hat{b}_1 = \mathrm{sign}(\, y_1 \,)$ is the ML detector.

(b) For $K = 2$, simulate the system using the same detector as in (a) and plot the BER of the two users versus E_b/N_0 for different $a = \left\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right\}$ and 0 dB $\leq E_b/N_0 \leq$ 15 dB. Explain the results that you observe.

(c) Prove that the joint ML detector is

$$\hat{\mathbf{b}} = \arg\max_{\mathbf{b}} \left[2\mathbf{y}^T\mathbf{b} - \mathbf{b}^T\mathbf{H}\mathbf{b}\right], \tag{4}$$

where $\hat{\mathbf{b}} = [\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_K]^T$ are the detected bits for the K users. (Hint: You first need to find the joint likelihood function $p(\mathbf{y} \mid \mathbf{b})$.)

(d) Simulate the system for $K = 3$ and $a = 1/4$ using the detectors in (a) and (c). Plot the BERs against E_b/N_0 for 0 dB $\leq E_b/N_0 \leq$ 15 dB.

(e) Since there are 2^K possible transmitted bit combinations, the computational complexity grows exponentially with K. Thus, different suboptimal detectors with less computational complexity have been proposed. Do a bit of research on multiuser detection to find at least two different suboptimal detectors. Implement one of these and plot the BER to compare its performance with the detectors in (a) and (c). Make sure to compare the complexity of your suboptimal detector to the ML scheme.

Please list all of the resources you used to answer this question including references to the subject notes, workshops and problem sets as well as any websites or textbooks etc.

7. **[15 marks] (Harder)** *The Viterbi Algorithm*

Consider molecular communication using On-Off Keying modulation where no molecules and N molecules are released at the beginning of a symbol interval to transmit bit "0" and "1", respectively. Assume that the two symbols are equiprobable. Let x_k and y_k denote the numbers of molecules released at the transmitter and received at the receiver in the k-th symbol interval. Due to the random movement of molecules, some molecules released in the k-th interval can arrive in later intervals, causing inter-symbol interference (ISI). The receiver also receives some molecules from a noise source in the environment. Assume that significant ISI is only from one previous symbol and y_k follows a Poisson distribution with mean given by

$$\bar{y}_k = x_k p_1 + x_{k-1} p_2 + n, \tag{5}$$

where $n = 1$, $p_1 = 0.15$, and $p_2 = 0.10$ are the expected number of noise molecules received in an interval and the probabilities that a molecule released from the transmitter at time zero arrives at the receiver within the first and second symbol intervals, respectively.

Assume that we receive $y_1, y_2, \ldots, y_{150}$ and that we wish the determine $x_1, x_2, \ldots, x_{150}$. Also assume that the first received symbol is not affected by ISI.

(a) Derive the maximum likelihood sequence detector for the system. If your detection involves a product of terms, can you reduce the computational complexity by transforming it to a summation?

(b) Now suppose you wish to implement the maximum likelihood sequence detector from (a) using the Viterbi algorithm. Sketch the trellis diagram for the first 4 symbol transmissions and specify the states and path lengths, i.e., cost metric functions.

(c) Write Matlab code to implement the Viterbi algorithm on a block of 250 received signals. Test your algorithm by repeatedly generating random $x_1, x_2, \ldots, x_{150}$ and then $y_1, y_2, \ldots, y_{150}$, running the Viterbi algorithm, and counting the number of errors. Plot the BER versus N for $50 \leq N \leq 350$.

Please list all of the resources you used to answer this question including references to the subject notes, workshops and problem sets as well as any websites or textbooks etc.