NPTEL — Linear Algebra, July–Oct 2022 (Repeat)

Week 5

Exercise Set 5 : Dimension of vector spaces

Submit solutions of the ANY TWO blue coloured *Exercises ONLY

Let K be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

***5.1** Let $n \in \mathbb{N}$, $n \geq 2$. Determine whether or not the vectors

(a) $(1,1,\ldots,1),(1,2,1,\ldots,1),\ldots,(1,\ldots,1,n)$ form a basis of $\mathbb{R}^n$ (resp. $\mathbb{Q}^n$).

(b) $(-(n-1),1,\ldots,1),(1,-(n-1),1,\ldots,1),\ldots,(1,\ldots,1,-(n-1))$ form a basis of $\mathbb{R}^n$ (resp. $\mathbb{Q}^n$).

5.2 Let K be a finite field with q elements.

(a) The multiples $m\cdot 1_K$, $m \in \mathbb{Z}$, form a subfield K' of K.

(b) There exists a smallest positive natural number p such that $p\cdot 1_K = 0$. Moreover, it is prime (and is called the characteristic of K — denoted by $\operatorname{Char} K$). The subfield $K' \subseteq K$ contains exactly p distinct elements $0, 1_K, \ldots, (p-1)1_K$.

(c) Show that $q = p^n$ with $n := \operatorname{Dim}_{K'} K$. (**Remark :** *The number of elements is a finite field is a power of a prime number.* Conversely, for a given prime-power q, there exists (essentially unique) field with q elements.)

5.3 Let $\omega \in \mathbb{R}_+^\times$ be a fixed positive real number. For $a \in \mathbb{R}$ and $\varphi \in \mathbb{R}$, let $f_{a,\varphi} : \mathbb{R} \to \mathbb{R}$ be the function defined by $t \mapsto a\sin(\omega t + \varphi)$ and let $W := \{f_{a,\varphi} \mid a, \varphi \in \mathbb{R}\}$. Then W is a $\mathbb{R}$-subspace of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$ of all $\mathbb{R}$-valued functions on $\mathbb{R}$.

(a) Find a $\mathbb{R}$-basis of the $\mathbb{R}$-subspace W. What is the dimension $\operatorname{Dim}_{\mathbb{R}} W$?

(**Hint :** The functions $t \mapsto \sin\omega t$ and $t \mapsto \cos\omega t = \sin(\omega t + \pi/2)$ form a basis of W. — **Remark :** Elements of W are called harmonic oscillations with the circular frequency ω.)

(b) Show that every $f \neq 0$ function in W has a unique representation

$$f(t) = a\sin(\omega t + \varphi)\,, \qquad a > 0 \quad \text{and} \quad 0 \leq \varphi < 2\pi\,.$$

(**Remark :** This unique a is called the amplitude and φ is called the phase angle of f. The zero function has the amplitude 0 and an arbitrary phase angle.)

(c) From the amplitudes and the phase angles of two harmonic oscillations f and g, compute the amplitudes and the phase angles of the functions $f \pm g$.

5.4 Let V be a K-vector space of dimension $n \in \mathbb{N}$.

(a) If $H_1,\ldots,H_r$ are hyper-planes in V, then show that $\operatorname{Dim}_K(H_1 \cap \cdots \cap H_r) \geq n - r$.

(b) If $U \subseteq V$ is a subspace of codimension r, then show that there exist r hyper-planes $H_1,\ldots,H_r$ in V such that $U = H_1 \cap \cdots \cap H_r$.

5.5 Let K be a field and let $a_0,\ldots,a_m \in K$, $a_m \neq 0$. Show that the subset

$$V(a_0,\ldots,a_m) := \{(x_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}} \mid a_0x_n + a_1x_{n+1} + \cdots + a_mx_{n+m} = 0 \ \text{ for all } \ n \in \mathbb{N}\}$$

is a subspace of $K^{\mathbb{N}}$ of the dimension m. (**Hint:** Let $\mathrm{e}_n := (\delta_{i,n})_{i\in\mathbb{N}}$, $n \in \mathbb{N}$ and $\mathbf{s} : K^{\mathbb{N}} \to K^{\mathbb{N}}$, $\mathrm{e}_n \mapsto \mathrm{e}_{n+1}$, $n \in \mathbb{N}$, denote the shift operator on $K^{\mathbb{N}}$. Then $(x_n)_{n\in\mathbb{N}} \in V(a_0,\ldots,a_m)$ if and only if $(x_n)_{n\in\mathbb{N}} \in \operatorname{Ker}\alpha(\mathbf{s})$. Moreover, the operator $\alpha(\mathbf{s}) : K^{\mathbb{N}} \to K^{\mathbb{N}}$ is surjective and the map $\operatorname{Ker}\alpha(\mathbf{s}) \to K^m$, $(x_n)_{n\in\mathbb{N}} \mapsto (x_0,x_1,\ldots,x_{m-1})$ is an isomorphism of K-vector spaces.

– **Remark:** We say that a sequence $(x_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ satisfy the (linear) recursion equation with (recursion) polynomial $\alpha(X) = a_0 + a_1X + \cdots + a_mX^m \in K[X]$ if the sequence $(x_n)_{n\in\mathbb{N}} \in V(a_0,\ldots,a_m)$. If K is algebraically closed (for example, if $K = \mathbb{C}$), then one can also find a K-basis of $V(a_0,\ldots,a_m)$ in by using the zeros of the polynomial $\alpha(A)$.)

****5.6** Let $x_1 = (a_{11},\ldots,a_{1n}),\ldots,x_n = (a_{n1},\ldots,a_{nn})$ be elements of $\mathbb{K}^n$ with

$$|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ji}| \quad \text{for all } i = 1,\ldots,n.$$

Show that $x_1,\ldots,x_n$ is a basis of $\mathbb{K}^n$.

(**Hint:** It is enough to show the linear independence of $x_1,\ldots,x_n$. If $b_1x_1 + \cdots + b_nx_n = 0$ with $|b_i| \leq 1$ for all i and $b_{i_0} = 1$ for some i_0. This already contradicts the give condition for i_0.)

5.7 Let V be a finite dimensional K-vector space and let U be a subspace of V. Let $u_1,\ldots,u_m$ be a basis of U and let $u_1,\ldots,u_m,u_{m+1},\ldots,u_n$ be an extended basis of V. Show that

$$x = a_1u_1 + \cdots + a_mu_m + b_{m+1}u_{m+1} + \cdots + b_nu_n \in V$$

is an element of U if and only if the coordinates $b_{m+1} = u^*_{m+1}(x),\ldots,b_n = u^*_n(x)$ of x with respect to the basis $u_1,\ldots,u_n$ of V are zero. (**Remark:** This is the most common method of characterizing the elements of a subspace.)

****5.8** Let $x_1,\ldots,x_n \in \mathbb{Z}^n$ be arbitrary vectors with integer components. For every $\lambda \in \mathbb{Q}\setminus\mathbb{Z}$, the vectors $x_1 + \lambda e_1,\ldots,x_n + \lambda e_n$ form a basis of $\mathbb{Q}^n$.

(**Hint:** Suppose $a_1(x_1 + \lambda e_1) + \cdots + a_n(x_n + \lambda e_n) = 0$ with $a_1,\ldots,a_n \in \mathbb{Z}$ and $\gcd(a_1,\ldots,a_n) = 1$ and use $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$ to contradict $\gcd(a_1,\ldots,a_n) = 1$.)

****5.9** Let K be a field with at least n elements, $n \in \mathbb{N}^*$ and V be a finite dimensional K-vector space. Let $U_1,\ldots,U_n$ be subspaces of V of equal dimension r and $u_{1i},\ldots,u_{ir}$ be a basis of U_i for $i = 1,\ldots,n$. Show that there exists $t := \operatorname{Dim}_K V - r$ vectors $w_1,\ldots,w_t \in V$ such that which simultaneously extend the given bases $u_{1i},\ldots,u_{ir}$ of U_i to a basis $u_{i1},\ldots,u_{ir},w_1,\ldots,w_t$ of V for every $i = 1,\ldots,n$. (**Hint** Use Exercise 2.2.).

5.10 Let K be a field and $F = a_0 + a_1X + \cdots + a_nX^n \in K[X]$ be a polynomial of degree $\deg F = n$, $n \in \mathbb{N}$. Suppose that the multiples $m \cdot 1_K$ are all $\neq 0$ for all $m \in \mathbb{N}^*$, i.e. $\operatorname{Char} K = 0$, see Exercise 5.2 (b). For example, $K = \mathbb{Q}$, $\mathbb{R}$ and $K = \mathbb{C}$ have this property. For pairwise distinct elements $\lambda_0,\ldots,\lambda_n \in K$, the polynomials $F(X-\lambda_0),\ldots,F(X-\lambda_n) \in K[X]_{n+1}$ form a K-basis of the K-vector space $K[X]_{n+1}$ of polynomials of degree $\leq n$ over K. In particular, the polynomials $(X-\lambda_0)^n,\ldots,(X-\lambda_n)^n$ form a basis of $K[X]_{n+1}$.

(**Hint:** Since $\operatorname{Dim}_K K[X]_{n+1} = n+1$ and hence it is enough to prove the linear independence of $F(X-\lambda_0),\ldots,F(X-\lambda_n)$ over K which is proved in Exercise 4.6 (b).)