C3 Examination 40 Marks

Question 1: (5 Marks)

a) If X_1, X_2 are iid, then show that for $t > 0$

$$P(|X_1 - X_2| > t) \leq 2P(|X_1| > t/2)$$

b) If $X_1, X_2, \ldots, X_n$ are independent with a common distribution function F. Then

$$2P(|X_1 + X_2 + \cdots + X_n| \geq t) \geq 1 - \exp(-n[1 - F(t) + F(-t)])$$

Question 2: (5 Marks)

Let $X_1, X_2, \ldots, X_n$ be 'n' independent random variables such that $E[X_i] = 0$, $V(X_i) = \sigma_i^2 < \infty$

Show that

$$P\left(\max_{1 \leq k \leq n} \left|\sum_{j=1}^{k} X_j\right| > \varepsilon\right) < \frac{1}{\varepsilon} \sum_{j=1}^{n} \sigma_j^2$$

hint: Markov inequality

Question 3 (5 marks)

a) Explain Central limit theorems for iid random variables.

b) Compute the characteristic function for Normal random varoable $N(0,1)$.

Question 4 (5 Marks)

a) Let $\{X_n\}$ be sequence of random variables and $X_n \rightarrow c$ (X_n convergee to c)
Write docon defeinition of (I) conveergence in probabeliry

(II) almost surely convergence

(III) convergence in (mean square) quadratic mean.

b) (I) Show that. Almost sure conversence i.e. $X_n \xrightarrow{a.s.} c$ implies convergence in probability, i.e. $X_n \xrightarrow{P.} c$

(II) show that convergence in mean square, i.e. $X_n \xrightarrow{m.s} c$ implies convergence in probability, i.e. $X_n \xrightarrow{P.} c$

Question 5 : (5 Marks)

a) Explain probability axioms

b) Define random variable and explain properties of distribution function.

Question 6 : (5 Marks)

a) Explain conditional probability and Bayes Rule

b) Explain conditional distribution of a random variable

c) if Two random variables X_1, X_2 are said to be independent ~~is~~, then what is joint distribution function?

Question 7 : (5 Marks)

Definition : Let $X_1, X_2, \ldots, X_n$ be sequence of random variables. F_n be cumulative distribution function of random variable X_n, $n \geq 0$. Then X_n converges to random variable X, in distribution

i.e. $X_n \xrightarrow{d} X$, if

$$① \text{ — } \lim_{n \to \infty} F_n(x) = F(x) \quad \text{for every } x \in C(F_0)$$

~~C(F) F is continuous at x~~

$$C(F_0) = \{x \in \mathbb{R} : F_0 \text{ is continuous at } x\}$$

① can be written as follows

$$② \quad \lim_{n \to \infty} P(X_n \leq x) = P(X \leq x)$$

Q.a) For $n \geq 1$, X_n is uniformly distributed $U_n = (0, \frac{1}{n})$

write down cummulative distribution function $F_n(x)$.

$F(x)$ is (cdf) distribution of degenerate random variable X

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Then show that $F_n(x) \xrightarrow{d} F(x)$

b) if $X_n \xrightarrow{P} X$ (X_n converges to X in probability)

then $X_n \xrightarrow{d} X$ (X_n converges to X in distribution)

(show it)

Question 8 5 Marks

Let $\{X_n\}$ be iid random variables with $E[X_n] = \mu$, $E[X_n^2] < \infty$.

$$S_0 = 0, \quad S_n \triangleq \frac{1}{n}\sum_{i=1}^{n} X_i$$

By strong law of large numbers $S_n \to \mu$ (a.s.)

Almost surely.

a) Write down iterative algorithm for 'S_n' in the form

$$S_{n+1} = S_n + a(n)\left[h(S_n) + M_{n+1}\right]$$

Define $a(n)$, $h(S_n)$, M_{n+1} suitaubly.

b) Consider an initially empty urn to which, either red or black ball, are added one at a time

Let y_n denote the number of red balls at time 'n'

and $x_n = \frac{y_n}{n}$.

$$y_{n+1} = y_n + \xi_{n+1}$$

$\xi_{n+1} = 1$ if the $(n+1)$ ball is red

$\xi_{n+1} = 0$ if the $(n+1)$ ball is black

write down iterative eequation for 'x_n'.

if one ball is drawn from urn, if that ball is red, red ball is added otherwise black ball is added.

$p(x_0)$: the probability of first ball is being red.

write $x_{n+1} = x_n + \frac{1}{n+1}\left[h(x_n) + M_{n+1}\right]$,

define $h(x_n)$ & M_{n+1}, suitably.