Exercise 1 : (6 Marks)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with

$$Pr(X_i = 1) = Pr(X_i = -1) = \frac{1}{2}$$

Let $S = \sum_{i=1}^{n} X_i$

For any $a > 0$, show that

a) $P(S \geq a) \leq e^{-a^2/2n}$

b) $P(|S| \geq a) \leq 2e^{-a^2/2n}$

Exercise 2: Let $Z_1, Z_2, \ldots, Z_N$ be real valued random variable & $\psi_{Z_i}(\lambda) \leq \lambda^2 v/2$

Show that

$$E[\max_{i=1,2,\ldots,N} Z_i] \leq \sqrt{2v \log N}$$

hint: 1) Compute $\exp\left(\lambda E\left(\max_{i=1,2,\ldots,N} Z_i\right)\right)$

and show that

$$\exp\left[\lambda E\left[\max_{i=1,2,\ldots,N} Z_i\right]\right] \leq N e^{\lambda^2 v/2}$$

(6 marks)

use Jensen's Inequality

Exercise 3 (6 Marks)

Image tossing a biased coin that lands heads with probability p. and let our hypothesis ~~to~~ be the one that always guesses tails. Then the true error rate is $R(h) = p$ and the empirical error rate $\hat{R}(h) = \hat{p}$, where $\hat{p}$ is the empirical probality of heads based on the training samples drawn iid. Show that

① $$P_r\left(|p-\hat{p}| \leq \sqrt{\frac{\log(2/\delta)}{2m}}\right) \geq 1-\delta$$

② if $\delta = 0.02$, $m = 500$, show that

$$P_r\left(|p-\hat{p}| \leq 0.048\right) \geq 0.98.$$

Use hoeffding inequality.

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Exercise 4:

① if X_1, X_2 are two independent random variables with d.f.'s F_1 and F_2, then the d.f.'s of $Z = X_1 + X_2$ is ?

② Jensen's Inequality:
If g is a convex function and $E[X] < \infty$,

then show that

$$g[E(X)] \leq E[g(X)]$$

③ $X_1, X_2, \ldots$ are independent random variables $X_i \in [0,1]$. Then show that for any $\varepsilon > 0$

$$P\left\{\left(\frac{1}{n} \sum_{i=1}^{n} (X_i - E(X_i))\right) \geq \varepsilon\right\} \leq e^{-2n\varepsilon^2}$$

Exercise 5 (6 Marks)

The characteristic function $f(t)$ of random variablea X with distribution function $F(x)$ and density function $f(x)$. is given by

$$h(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

a) Observe and show that $h(0) = 1$,

$$|h(t)| \leq 1, \quad h(-t) = \bar{h}(t)$$

where $\bar{h}(t)$ is complex conjugate

b) The moment generating function

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Is $M(t)$ always exists? if yes why?

if No, why?

c) Compute characteristic function of Normal random variable $N(0,1)$